X

PHOTON WAVE FUNCTION

BY

Iwo Bialynicki-Birula
Center for Theoretical Physics, Polish Academy of Sciences
Al. Lotników 32, 02-668 Warsaw, Poland
and
Rochester Theory Center for Optical Science and Engineering
University of Rochester, Rochester, NY 14627, USA

CONTENTS

3

§1.   Introduction

Photon wave function is a controversial concept. Controversies stem from the fact that photon wave functions can not have all the properties of the Schrödinger wave functions of nonrelativistic wave mechanics. Insistence on those properties that, owing to peculiarities of photon dynamics, cannot be rendered, led some physicists to the extreme opinion that the photon wave function does not exist. I reject such a fundamentalist point of view in favor of a more pragmatic approach. In my view, the photon wave function exists as long as it can be precisely defined and made useful. Many authors whose papers are quoted in this review share the same opinion and had no reservations about using the name the photon wave function when referring to a complex vector-function of space coordinates $\mathbf{r}$ and time t that adequately describes the quantum state of a single photon.

The notion of the photon wave function is certainly not new, but strangely enough it has never been systematically and fully explored. Some textbooks on quantum mechanics start the introduction to quantum theory with a discussion of photon polarization measurements (cf., for example Dirac [1958], Baym [1969], Lipkin [1973], Cohen-Tannoudji, Diu and Laloë [1977]), but in all these expositions a complete photon wave function never takes on a specific mathematical form. Even Dirac who writes "The essential point is the association of each of the translational states of the photon with one of the wave functions of ordinary wave optics", never expresses this association in an explicit form. In this context he also uses the now famous phrase: "Each photon interferes only with itself" which implies the existence of photon wave functions whose superposition leads to interference phenomena.

In the textbook analysis of polarization, only simple prototype two-component wave functions are used to describe various polarization states of the photon and with their help the preparation and the measurement of polarization is analyzed. However, it is not explained, why a wave function should not be used to describe also the "translational states of the photon" mentioned by Dirac. After such a heuristic introduction to quantum theory, the authors go on to the study of massive particles and if they ever return to quantum theory of photons it is always within the formalism of second quantization with creation and annihilation operators. In some textbooks (cf., for example, Bohm [1954], Power [1964]) one may even find statements that completely negate the possibility of introducing a wave function for the photon.

A study of the photon wave function should be preceded by an explanation what is the photon and why a description of the photon in terms of a wave func-

tion must exist. According to modern quantum field theory, photons, together with all other particles (and also quasiparticles, phonons, excitons, plasmons, etc.), are the *quantum excitations* of a field. In the case of photons, these are the excitations of the electromagnetic field. The lowest field excitation of a *given type* corresponds to one photon and higher field excitations involve more than one photon. This concept of a photon (called the modern photon in a tutorial review by Kidd, Ardini and Anton [1989]) enables one to use the photon wave function not only to describe quantum states of an excitation of the free field but also of the electromagnetic field interacting with a medium. Conceptually, the difference between free space and a medium is not essential since the physical vacuum is like a polarizable medium. It is filled with all the virtual pairs — zero point excitations of charged quantum fields. Therefore, even in free space, photons can be also viewed as the excitations of the vacuum made mostly of virtual electron-positron pairs (Bialynicki-Birula [1963], Bjorken [1963]).

Even though, in principle, all particles can be treated as field excitations, photons are much different from massive particles. They are also different from massless neutrinos since the photon number does not obey a conservation law. There are problems with the photon localization and as a result the position operator for the photon is ill-defined, but the similarities between photons and other quantum particles are so ample that the introduction of the photon wave function seems to be fully justified and even necessary in order to achieve a complete unification of our description of all particles.

1.1.   COORDINATE VS. MOMENTUM REPRESENTATION

In nonrelativistic quantum mechanics the term *coordinate representation* is used to denote the representation in which the wave function $\psi(\mathbf{r})$ is defined as a projection of the state vector $|\psi\rangle$ on the eigenstates $|\mathbf{r}\rangle$ of the components $\hat{x}, \hat{y}$, and $\hat{z}$ of the position operator $\hat{\mathbf{r}}$,

$$\psi(\mathbf{r}) = \langle \mathbf{r}|\psi\rangle. \tag{1.1}$$

The wave function in coordinate representation, therefore, becomes automatically a function of the eigenvalues of the position operator $\hat{\mathbf{r}}$. The position operators act on the wave function simply through a multiplication. In quantum mechanics of photons this approach does not work due to difficulties with the definition of the photon position operator (cf. §8). One may still, however, introduce functions of the coordinate vector $\mathbf{r}$ to describe quantum states of the photon. By adopting this less stringent point of view that does not tie the wave function in coordinate representation with the formula (1.1), one avoids

the consequences of the nonexistence of the photon position operator $\hat{\mathbf{r}}$. In principle, any function of $\mathbf{r}$ that adequately describes photon states may be called a photon wave function in coordinate representation and it is a matter of taste and convenience which one to use. It should be pointed out that in a relativistic quantum theory, even for particles with nonvanishing rest mass, the position operator and the localization associated with it do not live up to our nonrelativistic expectations. The differences in localization of photons and, say electrons, are more quantitative then qualitative since they amount to the "spilling of the wave function" beyond the localization region governed by a power law versus an exponential decay.

The photon wave function in *momentum representation* has not stirred any controversy since the photon momentum operator $\hat{\mathbf{p}}$ is well defined. Its existence, as the generator of translations, follows directly from the general theory of representations of the Poincaré group developed by Wigner [1939]. It has always been taken for granted by all physicists working in relativistic quantum electrodynamics that the notion of the photon wave function in momentum representation is well founded. Such wave functions describing initial and final states of photons appear in all formulas for transition amplitudes in the S-matrix theory of scattering phenomena (cf., for example, Schweber [1961], Akhiezer and Berestetskii [1965], Bialynicki-Birula and Bialynicka-Birula [1975], Cohen-Tannoudji, Dupont-Roc and Grynberg [1989]). Thus, one may safely assert that the photon wave function in momentum representation is a well defined and fully established object.

1.2. PHASE REPRESENTATION

The photon wave functions discussed in this review are distinct from the *one-mode* wave functions that have been introduced in the past (London [1927], Bialynicki-Birula and Bialynicka-Birula [1976], Pegg and Barnett [1988]) to describe *multi-photon states*. These functions depend on the phase φ of the field and were called the wave functions in the phase representation by Bialynicki-Birula and Bialynicka-Birula [1976] The wave functions $\Psi(\varphi)$ characterize quantum states of a *selected mode* of the quantized electromagnetic field and, in general, they describe a superposition of states with different numbers of photons. All spatial characteristics of these states are contained in the mode function that defines the selected mode of the electromagnetic field. One-mode wave functions $\Psi(\varphi)$ describe properties of multi-photon states of the quantized electromagnetic field with all photons being in the same quantum mechanical state. In contrast, the photon wave function in the coordinate representation can be identified with the mode func-

tion itself (cf. §13). It describes a state of a *single* photon and not a state of the quantized field.

1.3. LANDAU-PEIERLS WAVE FUNCTION

The concept of the photon wave function in coordinate representation was introduced for the first time by Landau and Peierls [1930]. The same function has been independently rediscovered more recently by Cook [1982a], Cook [1982b], and Inagaki [1994]. The Landau-Peierls proposal has not been met with great enthusiasm since their wave function is a highly nonlocal object.

The nonlocality of the Landau-Peierls wave function is introduced by operating on the local electromagnetic field with the integral operator $(-\Delta)^{-1/4}$,

$$((-\Delta)^{-1/4}f)(\mathbf{r}) = \pi \int \frac{d^3r'}{(2\pi|\mathbf{r}-\mathbf{r}'|)^{5/2}} f(\mathbf{r}'). \tag{1.2}$$

This integral operator corresponds to a division by $\sqrt{|\mathbf{k}|}$ of the Fourier transform and it changes the dimension of the wave function from L^{-2}, characteristic of the electromagnetic field, to $L^{-3/2}$. Therefore the modulus squared of the Landau-Peierls wave function has the right dimensionality to be interpreted as a probability density to find a photon. In particular, these wave functions can be normalized to one with the standard definition of the norm since the integral of the modulus squared of the wave function is dimensionless. However, as has been already noted by Pauli [1933], despite its right dimensionality the nonlocal wave function has serious drawbacks. First, it does not transform under Lorentz transformations as a tensor field or any other geometric object. Second, a nonlocal wave function taken at a point in one coordinate system depends on the values of this wave function in *all of space* in another coordinate system. Third, the probability density defined with the use of a nonlocal wave function does not correspond to the probability of interaction of localized charges with the electromagnetic field. The vanishing of the wave function at a definite point, in Pauli's words (Pauli [1933]), has "no direct physical significance" because the electromagnetic field does act on charges at the points where the probability to find a photon vanishes. The Landau-Peierls wave functions can not be used as primary objects in the presence of a medium since one is unable to impose proper boundary conditions on such nonlocal objects. These functions can be introduced, if one wishes so, as secondary objects related to the local wave function by a nonlocal transformation (§5). Scalar products and expectation values look simpler when they are expressed in terms of the Landau-Peierls wave functions but that is, perhaps, their only advantage.

1.4. RIEMANN-SILBERSTEIN WAVE FUNCTION

The mathematical object that fully deserves the name of the photon wave function can be traced back to a complexified form of Maxwell's equations that was known already at the turn of the century. The earliest reference is the second volume of the lecture notes on the differential equations of mathematical physics by Riemann that were edited and published by Weber [1901]. Various applications of this form of Maxwell's equations were given by Silberstein [1907a], Silberstein [1907b], Silberstein [1914] and Bateman [1915] in the framework of classical physics.

   The complex form of Maxwell's equations is obtained by multiplying the first pair of these equations

$$\partial_t \mathbf{D}(\mathbf{r}, t) = \nabla \times \mathbf{H}(\mathbf{r}, t), \quad \nabla \cdot \mathbf{D}(\mathbf{r}, t) = 0, \tag{1.3}$$

by the imaginary unit and then by subtracting from it the second pair

$$\partial_t \mathbf{B}(\mathbf{r}, t) = -\nabla \times \mathbf{E}(\mathbf{r}, t), \quad \nabla \cdot \mathbf{B}(\mathbf{r}, t) = 0. \tag{1.4}$$

In the SI units that are used here the vectors $\mathbf{D}$ and $\mathbf{B}$ have different dimensions and prior to subtraction one must equalize the dimensions of both terms. The resulting equations in empty space are

$$i \partial_t \mathbf{F}(\mathbf{r}, t) = c \nabla \times \mathbf{F}(\mathbf{r}, t), \tag{1.5}$$

$$\nabla \cdot \mathbf{F}(\mathbf{r}, t) = 0, \tag{1.6}$$

where

$$\mathbf{F}(\mathbf{r}, t) = \left(\frac{\mathbf{D}(\mathbf{r}, t)}{\sqrt{2\epsilon_0}} + i \frac{\mathbf{B}(\mathbf{r}, t)}{\sqrt{2\mu_0}} \right), \tag{1.7}$$

and $c = 1/\sqrt{\epsilon_0 \mu_0}$. Around the year 1930 Majorana (unpublished notes quoted by Mignani, Recami and Baldo [1974]) arrived at the same complex vector exploring the analogy between the Dirac equation and the Maxwell equations. Kramers [1938] made an extensive use of this vector in his treatment of quantum radiation theory. This vector is also a natural object to use in the quaternionic formulation of Maxwell's theory (Silberstein [1914]). With the advent of spinor calculus that superseded the quaternionic calculus, the transformation properties of the Riemann-Silberstein vector have become even more transparent. When Maxwell's equations were cast into the spinor form (Laporte and Uhlenbeck [1931], Oppenheimer [1931]), this vector turned into a symmetric second-rank spinor.

The use of the Riemann-Silberstein vector as the wave function of the photon has been advocated by Oppenheimer [1931], Molière [1949], Good [1957], Bialynicki-Birula [1994], Sipe [1995], and Bialynicki-Birula [1996a].

It has already been noticed by Silberstein [1907a] that the important dynamical quantities associated with the electromagnetic field: the energy density and the Poynting vector can be represented as *bilinear expressions* built from the complex vector $\mathbf{F}$. Using modern terminology, one would say that the formulas for the energy E, momentum $\mathbf{P}$, angular momentum $\mathbf{M}$, and the moment of energy $\mathbf{N}$ of the electromagnetic field look like quantum-mechanical expectation values

$$E \;=\; \int d^3r\, \mathbf{F}^* \cdot \mathbf{F}, \tag{1.8}$$

$$\mathbf{P} \;=\; \frac{1}{2ic} \int d^3r\, \mathbf{F}^* \times \mathbf{F}, \tag{1.9}$$

$$\mathbf{M} \;=\; \frac{1}{2ic} \int d^3r\, \mathbf{r} \times (\mathbf{F}^* \times \mathbf{F}), \tag{1.10}$$

$$\mathbf{N} \;=\; \int d^3r\, \mathbf{r}\, (\mathbf{F}^* \cdot \mathbf{F}), \tag{1.11}$$

evaluated in a state described by the wave function $\mathbf{F}$. All these quantities are invariant under the multiplication of $\mathbf{F}$ by a phase factor $\exp(i\alpha)$. Such a multiplication results in the so called duality rotation (Misner and Wheeler [1957]) of the field vectors

$$\mathbf{D}'/\sqrt{\epsilon_0} = \cos\alpha\, \mathbf{D}/\sqrt{\epsilon_0} - \sin\alpha\, \mathbf{B}/\sqrt{\mu_0}, \tag{1.12}$$

$$\mathbf{B}'/\sqrt{\mu_0} = \cos\alpha\, \mathbf{B}/\sqrt{\mu_0} + \sin\alpha\, \mathbf{D}/\sqrt{\epsilon_0}. \tag{1.13}$$

The coupling of the electromagnetic field with charges fixes the phase α but for a free photon one has the same complete freedom in choosing the overall phase of the photon wave function as in standard wave mechanics of massive particles.

The Riemann-Silberstein vector has also many other properties that one would associate with a one-photon wave function, except for a somewhat modified probabilistic interpretation. Insistence on exactly the same form of the expressions for transition probabilities as in nonrelativistic wave mechanics leads back to the Landau-Peierls wave function with its highly nonlocal transformation properties.

§2.   Wave equation for photons

The wave equation for the photon is taken to be the complexified form (1.5) of Maxwell's equations. In order to justify this choice, one may show (cf. §4) that the Fourier decomposition of the solutions of this wave equation leads to the same photon wave functions in momentum representation that can be introduced without any reference to a wave equation directly from the general theory of representations of the Poincaré group (Bargmann and Wigner [1948], Lomont and Moses [1962]). There is also a heuristic argument indicating that eq. (1.5) is the right choice. Namely, the wave equation (1.5) can be written in the same form as the Weyl equation for the neutrino wave function. As a matter of fact, all wave equations for massless particles with arbitrary spin can be cast into the same form (cf. §12).

2.1.   WAVE EQUATION FOR PHOTONS IN FREE SPACE

In order to see a correspondence between Maxwell's equations and quantum mechanical wave equations, one may follow Oppenheimer [1931] and Molière [1949] and rewrite (1.5) with the use of the spin-1 matrices s_x, s_y, s_z well known from quantum mechanics (see, for example, Schiff [1968]). The matrices that will be used here are in a different representation from the one usually used in quantum mechanics since they act on the Cartesian vector components of the wave function and not on the components labeled by the eigenvalues of s_z. That is the reason why the matrix s_z is not diagonal.

$$s_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \; s_y = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix}, \; s_z = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{2.1}$$

Eq. (1.5) can be written in terms of these spin matrices if the following conversion rule from vector notation to matrix notation is applied

$$\mathbf{a} \times \mathbf{b} = -i(\mathbf{a} \cdot \mathbf{s})\mathbf{b}. \tag{2.2}$$

The resulting equation

$$i\hbar\partial_t\mathbf{F}(\mathbf{r}, t) = c\Big(\mathbf{s} \cdot \frac{\hbar}{i}\nabla\Big)\mathbf{F}(\mathbf{r}, t), \tag{2.3}$$

is of a Schrödinger type but with a different Hamiltonian. The divergence condition (1.6) can also be expressed in terms of spin matrices either as

$$(\mathbf{s} \cdot \nabla)s_j \, \mathbf{F} = \nabla_j \, \mathbf{F}, \tag{2.4}$$

or equivalently (Pryce [1948]) as

$$\Big(\mathbf{s}\cdot\nabla\Big)^2\mathbf{F} = \Delta\mathbf{F}. \tag{2.5}$$

The form (2.3) of the Maxwell equations compares directly with the Weyl equation for neutrinos (Weyl [1929])

$$i\hbar\partial_t\phi(\mathbf{r},t) = c\Big(\boldsymbol{\sigma}\cdot\frac{\hbar}{i}\nabla\Big)\phi(\mathbf{r},t). \tag{2.6}$$

Eq. (2.6) differs from (2.3) only in having the Pauli matrices, appropriate for spin-1/2 particles, instead of the spin-1 matrices that are appropriate for photons. Of course, one may cancel the factors of $\hbar$ appearing on both sides of eqs. (2.3) and (2.6), but their presence makes the connection with quantum mechanics more transparent.

Some authors (Oppenheimer [1931], Ohmura [1956], Moses [1959]) introduced a different, though equivalent, form of the photon wave equation for a *four-component* wave function. The inclusion of the forth component enables one to incorporate the divergence condition in a natural way. This approach is directly related to the spinorial representation of the photon wave function and will be discussed in §12.

In quantum mechanics the stationary solutions of the wave equation play a distinguished role. They are the building blocks from which all solutions can be constructed. Stationary solutions of the wave equation are obtained by separating the time variable and solving the resulting eigenvalue problem. The eigenvalue equation resulting from the photon wave equation (2.3) is

$$c\Big(\mathbf{s}\cdot\frac{\hbar}{i}\nabla\Big)\mathbf{F}(\mathbf{r}) = \hbar\omega\mathbf{F}(\mathbf{r}). \tag{2.7}$$

Assuming that the photon energy $\hbar\omega$ is positive, one reads from (2.7) that the projection of the spin on the direction of momentum (helicity) is positive. It can easily be checked that one can reverse this sign by changing i into $-i$ in the definition (1.7) of the Riemann-Silberstein vector. Thus, the choice of sign in the definition of this vector is equivalent to choosing positive or negative helicity, corresponding to left-handed or right-handed circular polarization. In order to account for both helicity states of the photons, one has to consider both vectors; one may call them $\mathbf{F}_+$ and $\mathbf{F}_-$. This doubling of the vectors $\mathbf{F}$ has already been considered by Silberstein [1914] in the context of the classical Maxwell equations.

Of course, if one is only interested in translating the Maxwell equations into a complex form, one can restrict oneself to either $\mathbf{F}_+$ or $\mathbf{F}_-$. In both cases one obtains a one-to-one correspondence between the real field vectors $\mathbf{D}$ and $\mathbf{B}$ and their complex combination $\mathbf{F}_\pm$. However, to have a bona fide photon wave function one must be able to superpose different helicity states without changing the sign of the energy (frequency). This can only be done when both helicities are described by different components of *the same wave function* as in the theory of spin-1/2 particles. One can see even more clearly the need to use both complex combinations when one deals with the propagation of photons in a medium.

2.2. WAVE EQUATION FOR PHOTONS IN A MEDIUM

In free space the two vectors $\mathbf{F}_\pm$ satisfy two separate wave equations

$$i\partial_t \mathbf{F}_\pm(\mathbf{r}, t) = \pm c\nabla \times \mathbf{F}_\pm(\mathbf{r}, t). \tag{2.8}$$

In a homogeneous medium, using the values of ϵ and μ for the medium in the definition of the vectors $\mathbf{F}_\pm$,

$$\mathbf{F}_\pm(\mathbf{r}, t) = \left(\frac{\mathbf{D}(\mathbf{r}, t)}{\sqrt{2\epsilon}} \pm i\frac{\mathbf{B}(\mathbf{r}, t)}{\sqrt{2\mu}}\right), \tag{2.9}$$

one also obtains two separate wave equations. The new vectors $\mathbf{F}_\pm$ are linear combinations of the free-space vectors $\mathbf{F}_\pm^0$,

$$\mathbf{F}_+ = \frac{1}{2}\left[\left(\sqrt{\frac{\epsilon_0}{\epsilon}} + \sqrt{\frac{\mu_0}{\mu}}\right)\mathbf{F}_+^0 + \left(\sqrt{\frac{\epsilon_0}{\epsilon}} - \sqrt{\frac{\mu_0}{\mu}}\right)\mathbf{F}_-^0\right], \tag{2.10}$$

$$\mathbf{F}_- = \frac{1}{2}\left[\left(\sqrt{\frac{\epsilon_0}{\epsilon}} - \sqrt{\frac{\mu_0}{\mu}}\right)\mathbf{F}_+^0 + \left(\sqrt{\frac{\epsilon_0}{\epsilon}} + \sqrt{\frac{\mu_0}{\mu}}\right)\mathbf{F}_-^0\right]. \tag{2.11}$$

Thus, the positive and negative helicity states in a medium are certain linear superpositions of such states in free space. The necessity to form linear superpositions of both helicity states shows up even more forcefully in an inhomogeneous medium, because then it is not possible to split the wave equations into two independent sets. For a linear, time-independent, isotropic medium, characterized by space-dependent permittivity and permeability, one obtains the following coupled set of wave equations

$$i\partial_t \mathbf{F}_+(\mathbf{r}, t) = v(\mathbf{r})\left(\nabla \times \mathbf{F}_+(\mathbf{r}, t)\right) \tag{2.12}$$

$$- \frac{1}{2v(\mathbf{r})}\mathbf{F}_+(\mathbf{r},t) \times \nabla v(\mathbf{r}) - \frac{1}{2h(\mathbf{r})}\mathbf{F}_-(\mathbf{r},t) \times \nabla h(\mathbf{r})),$$

$$i\partial_t \mathbf{F}_-(\mathbf{r},t) = -v(\mathbf{r})\big(\nabla \times \mathbf{F}_-(\mathbf{r},t)\big) \tag{2.13}$$

$$- \frac{1}{2v(\mathbf{r})}\mathbf{F}_-(\mathbf{r},t) \times \nabla v(\mathbf{r}) - \frac{1}{2h(\mathbf{r})}\mathbf{F}_+(\mathbf{r},t) \times \nabla h(\mathbf{r})),$$

where $\mathbf{F}_\pm(\mathbf{r},t)$ are built with the values of $\epsilon(\mathbf{r})$ and $\mu(\mathbf{r})$ in the medium, $v(\mathbf{r}) = 1/\sqrt{\epsilon(\mathbf{r})\mu(\mathbf{r})}$ is the value of the speed of light in the medium, and $h(\mathbf{r}) = \sqrt{\mu(\mathbf{r})/\epsilon(\mathbf{r})}$ is the "resistance of the medium" (the sole justification for the use of this name is the right dimensionality of Ohm). The divergence condition (1.6) in an inhomogeneous medium takes on the form,

$$\nabla \cdot \mathbf{F}_+(\mathbf{r},t) = \frac{1}{2v(\mathbf{r})}\mathbf{F}_+(\mathbf{r},t) \cdot \nabla v(\mathbf{r}) + \frac{1}{2h(\mathbf{r})}\mathbf{F}_-(\mathbf{r},t) \cdot \nabla h(\mathbf{r}), \tag{2.14}$$

$$\nabla \cdot \mathbf{F}_-(\mathbf{r},t) = \frac{1}{2v(\mathbf{r})}\mathbf{F}_-(\mathbf{r},t) \cdot \nabla v(\mathbf{r}) + \frac{1}{2h(\mathbf{r})}\mathbf{F}_+(\mathbf{r},t) \cdot \nabla h(\mathbf{r}). \tag{2.15}$$

The quantum-mechanical form of eqs. (2.12) and (2.13) is

$$i\hbar\partial_t \mathbf{F}_+(\mathbf{r},t) = \sqrt{v(\mathbf{r})}\big(\mathbf{s} \cdot \frac{\hbar}{i}\nabla\big)\sqrt{v(\mathbf{r})}\mathbf{F}_+(\mathbf{r},t)$$

$$- i\hbar\frac{v(\mathbf{r})}{2h(\mathbf{r})}\big(\mathbf{s} \cdot \nabla h(\mathbf{r})\big)\mathbf{F}_-(\mathbf{r},t), \tag{2.16}$$

$$i\hbar\partial_t \mathbf{F}_-(\mathbf{r},t) = \sqrt{v(\mathbf{r})}\big(\mathbf{s} \cdot \frac{\hbar}{i}\nabla\big)\sqrt{v(\mathbf{r})}\mathbf{F}_-(\mathbf{r},t)$$

$$+ i\hbar\frac{v(\mathbf{r})}{2h(\mathbf{r})}\big(\mathbf{s} \cdot \nabla h(\mathbf{r})\big)\mathbf{F}_+(\mathbf{r},t). \tag{2.17}$$

In view of the coupling in the evolution equations (2.16) and (2.17) between the vectors $\mathbf{F}_+$ and $\mathbf{F}_-$, one has to combine them together to form one wave function $\mathcal{F}$ with six components

$$\mathcal{F} = \left[\begin{array}{c} \mathbf{F}_+ \\ \mathbf{F}_- \end{array} \right]. \tag{2.18}$$

The wave equation for this function can be written in a compact form

$$i\hbar\partial_t \mathcal{F}(\mathbf{r},t) = \sqrt{v(\mathbf{r})}\rho_3\big(\mathbf{s} \cdot \frac{\hbar}{i}\nabla\big)\sqrt{v(\mathbf{r})}\mathcal{F}(\mathbf{r},t)$$

$$+ \hbar\frac{v(\mathbf{r})}{2h(\mathbf{r})}\rho_2\big(\mathbf{s} \cdot \nabla h\big)\mathcal{F}(\mathbf{r},t), \tag{2.19}$$

where the spin matrices s_i operate separately on upper and lower components

$$s_i \mathcal{F} = \left[\begin{array}{c} s_i \mathbf{F}_+ \\ s_i \mathbf{F}_- \end{array} \right], \tag{2.20}$$

and the three Pauli matrices ρ_i act on $\mathcal{F}$ as follows

$$\rho_1 \mathcal{F} = \left[\begin{array}{c} \mathbf{F}_- \\ \mathbf{F}_+ \end{array} \right], \quad \rho_2 \mathcal{F} = \left[\begin{array}{c} -i\mathbf{F}_- \\ i\mathbf{F}_+ \end{array} \right], \quad \rho_3 \mathcal{F} = \left[\begin{array}{c} \mathbf{F}_+ \\ -\mathbf{F}_- \end{array} \right]. \tag{2.21}$$

The divergence conditions (2.14) and (2.15) can also be written in this compact form

$$\nabla \cdot \mathcal{F}(\mathbf{r}, t) = \frac{1}{2v(\mathbf{r})} \mathcal{F}(\mathbf{r}, t) \cdot \nabla v(\mathbf{r}) + \rho_1 \frac{1}{2h(\mathbf{r})} \mathcal{F}(\mathbf{r}, t) \cdot \nabla h(\mathbf{r}). \tag{2.22}$$

One would not be able to write a linear wave equation for an inhomogeneous medium in terms of just one three-dimensional complex vector, without doubling the number of components. Note that the speed of light v may vary without causing necessarily the mixing of helicities. This happens, for example, in the gravitational field (cf. §11). It is only the space-dependent resistance $h(\mathbf{r})$ that causes mixing.

It is worth stressing that the study of the propagation of photons in an inhomogeneous medium separates clearly local wave functions from nonlocal ones. In free space there are many wave functions satisfying the same set of equations. For example, differentiations of the wave functions or integral operations of the type (1.2) do not change the form of these equations. The essential difference between various wave functions shows up forcefully in the study of the wave equation in an inhomogeneous medium, or in curved space (§11). All previous studies, except Bialynicki-Birula [1994], were restricted to propagation in *free space* and this very important point was completely missed. The photon wave equations in an inhomogeneous medium are not very simple but that is due, perhaps, to a phenomenological character of macroscopic electrodynamics. The propagation of a photon in a medium is a succession of absorptions and subsequent emissions of the photon by the charges that form the medium. The number of photons of a given helicity is, in general, not conserved in these processes and that accounts for all the complications. The photon wave equations in an inhomogeneous medium is describing in actual fact a propagation of some collective excitations of the whole system and not just of free photons.

2.3. ANALOGY WITH THE DIRAC EQUATION

The analogy with the relativistic electron theory, mentioned in the Introduction, becomes the closest when the photon wave equation is compared with the Dirac equation written in the chiral representation of the Dirac matrices. In this representation the bispinor is made of two relativistic spinors

$$\psi(\mathbf{r}, t) = \begin{bmatrix} \phi_A(\mathbf{r}, t) \\ \chi^{A'}(\mathbf{r}, t) \end{bmatrix}, \tag{2.23}$$

and the Dirac equation may be viewed as two Weyl equations coupled by the mass term

$$i\hbar\partial_t\phi(\mathbf{r}, t) = \ \ c\big(\boldsymbol{\sigma}\cdot\frac{\hbar}{i}\nabla\big)\phi(\mathbf{r}, t) + mc^2\chi(\mathbf{r}, t), \tag{2.24}$$

$$i\hbar\partial_t\chi(\mathbf{r}, t) = -c\big(\boldsymbol{\sigma}\cdot\frac{\hbar}{i}\nabla\big)\chi(\mathbf{r}, t) + mc^2\phi(\mathbf{r}, t). \tag{2.25}$$

These equations are analogous to eqs. (2.16) and (2.17) for the photon wave function. For photons, the role of the mass term is played by the inhomogeneity of the medium.

§3.   Photon wave function in coordinate representation

Despite a formal similarity between the wave equations for the photon (2.16) and (2.17) and for the electron (2.24) and (2.25), there is an important difference. Photons, unlike the electrons, do not have antiparticles and this fact influences the form of solutions of the wave equation and their interpretation.

3.1. PHOTONS HAVE NO ANTIPARTICLES

Elementary, plane-wave solutions of relativistic wave equations in free space are of two types: they have positive or negative frequency,

$$\exp(-i\omega t + \mathbf{k}\cdot\mathbf{r}) \quad \text{or} \quad \exp(i\omega t - \mathbf{k}\cdot\mathbf{r}). \tag{3.1}$$

According to relativistic quantum mechanics, solutions with positive frequency correspond to particles and solutions with negative frequency correspond to antiparticles. Thus, positive and negative frequency parts of the same solution of the wave equation describe two different physical entities: particle and antiparticle.

Photons do not have antiparticles, or to put it differently, antiphotons are identical with photons. Hence, the information carried by the negative frequency solutions must be the same as the information already contained in the positive frequency solution. Therefore, one may completely disregard the negative frequency part as redundant. An alternative method is to keep also the negative frequency part but to impose an additional condition on the solutions of the wave equation. This condition states that the operation of particle-antiparticle conjugation

$$\mathcal{F}^c(\mathbf{r}, t) = \rho_1 \mathcal{F}^*(\mathbf{r}, t),\tag{3.2}$$

leaves the function $\mathcal{F}$ invariant

$$\mathcal{F}^c(\mathbf{r}, t) = \mathcal{F}(\mathbf{r}, t).\tag{3.3}$$

This condition is compatible with the evolution equation (2.19) and it eliminates the unwanted degrees of freedom (cf. Bialynicki-Birula [1994]). Eq. (3.3) is automatically satisfied if the wave function is constructed according to the definition (2.18). This follows from the fact that $\mathbf{F}_+$ and $\mathbf{F}_-$ are complex conjugate to each other. The information carried by the six-component function $\mathcal{F}$ satisfying the condition (3.3) is contained in its positive energy part and is the same as that carried by the initial Riemann-Silberstein vector $\mathbf{F}$. It follows from (3.3) that the negative frequency part $\mathcal{F}^{(-)}$ can always be obtained by complex conjugation and by an interchange of the upper and lower components of the positive frequency part $\mathcal{F}^{(+)}$,

$$\mathcal{F}^{(-)}(\mathbf{r}, t) = \rho_1 \mathcal{F}^{(+)*}(\mathbf{r}, t).\tag{3.4}$$

In this review, I shall use the symbol Ψ to denote the properly normalized, *positive energy* (positive frequency) part of the function $\mathcal{F}$

$$\Psi(\mathbf{r}, t) = \mathcal{F}^{(+)}(\mathbf{r}, t).\tag{3.5}$$

This is the true *photon wave function*. Proper normalization of the photon wave function is essential for its probabilistic interpretation and is discussed in §5. Note that the function Ψ carries the same amount of information as the original Riemann-Silberstein vector since Ψ can be constructed from $\mathbf{F}$ by splitting this vector into positive and negative frequency parts and then using the first part as the upper components of Ψ and the complex conjugate of the second part as the lower components,

$$\Psi(\mathbf{r}, t) = \begin{bmatrix} \mathbf{F}^{(+)}(\mathbf{r}, t) \\ \mathbf{F}^{(-)*}(\mathbf{r}, t) \end{bmatrix}.\tag{3.6}$$

The positive frequency part of the solutions of wave equations is a well known concept also in classical electromagnetic theory where it is called the analytic signal (Born and Wolf [1980], Mandel and Wolf [1995]).

3.2. TRANSFORMATION PROPERTIES OF THE PHOTON WAVE FUNCTION IN COORDINATE REPRESENTATION

In free space, the components of the electromagnetic field form a tensor and that allows one to establish the transformation properties of $\mathcal{F}$. Transformation properties of the photon wave function Ψ are the same as those of $\mathcal{F}$. Under rotations, the upper and the lower half of Ψ transform as three-dimensional vector fields. Under Lorentz transformations, the upper and the lower part also transform independently and the corresponding rules can be inferred directly from classical electrodynamics (cf., for example, Jackson [1975]). Under the Lorentz transformation characterized by the velocity $\mathbf{v}$, the vectors $\mathbf{F}_\pm$ change as follows

$$\mathbf{F}'_\pm = \gamma\big(\mathbf{F} \mp i\frac{\mathbf{v} \times \mathbf{F}}{c}\big) - \frac{\gamma^2}{\gamma+1}\frac{\mathbf{v}(\mathbf{v}\cdot\mathbf{F})}{c^2}, \tag{3.7}$$

where γ is the standard relativistic factor $\gamma = \sqrt{1 - v^2/c^2}$. One may check that this transformation preserves the square of these vectors,

$$(\mathbf{F}')^2 = (\mathbf{F})^2. \tag{3.8}$$

This is easily understood if one observes that $\mathbf{F}_\pm^2$ is a combination of the well-known scalar invariant $\mathcal{S} = (\epsilon_0\mathbf{E}^2 - \mathbf{B}^2/\mu_0)/2$ and the pseudoscalar invariant $\mathcal{P} = \sqrt{\epsilon_0/\mu_0}\mathbf{E}\cdot\mathbf{B}$ of the electromagnetic field

$$\mathbf{F}_\pm^2 = \mathcal{S} \pm i\mathcal{P}. \tag{3.9}$$

Thus, rotations and Lorentz transformations act on vectors $\mathbf{F}_\pm$ as elements of the orthogonal group in three dimensions (Kramers [1938] p. 429) leading to the following transformation properties of the wave function

$$\Psi'(\mathbf{r}',t') = \begin{bmatrix} C & 0 \\ 0 & C^* \end{bmatrix} \Psi(\mathbf{r},t), \tag{3.10}$$

where C is a three-dimensional, complex orthogonal matrix and C^* is its complex conjugate. The unification of rotations and Lorentz boosts into one complex

orthogonal transformation is even more transparent for infinitesimal transformations

$$\Psi' = \Psi + \big((i\delta\boldsymbol{\gamma} + \rho_3\delta\mathbf{v})\cdot\mathbf{s}\big)\Psi, \tag{3.11}$$

where $\delta\boldsymbol{\gamma}$ is the vector of an infinitesimal rotation. Under the space reflection $\mathbf{r} \rightarrow -\mathbf{r}$, the upper and lower part of Ψ do not transform independently but are interchanged because $\mathbf{D}$ and $\mathbf{B}$ transform as a vector and a pseudovector, respectively,

$$\Psi'(-\mathbf{r}, t) = \rho_1 \Psi(\mathbf{r}, t). \tag{3.12}$$

All these transformation properties can also be simply stated in terms of second rank spinor fields (cf. §12).

3.3. PHOTON HAMILTONIAN

The operator appearing on the right hand side of the evolution equation (2.19) for the photon wave function is the Hamiltonian operator $\hat{H}$ for the photon

$$\hat{H} = \sqrt{v(\mathbf{r})}\rho_3\big(\mathbf{s}\cdot\frac{\hbar}{i}\nabla\big)\sqrt{v(\mathbf{r})} + \hbar\frac{v(\mathbf{r})}{2h(\mathbf{r})}\rho_2\big(\mathbf{s}\cdot\nabla h(\mathbf{r})\big). \tag{3.13}$$

In free space this expression reduces to

$$\hat{H}_0 = c\rho_3\big(\mathbf{s}\cdot\frac{\hbar}{i}\nabla\big). \tag{3.14}$$

The formulas (3.13) and (3.14) define a Hermitian operator with continuous spectrum extending from $-\infty$ to ∞. Hermiticity is defined here with respect to the standard (mathematical) scalar product

$$(\Psi_1|\Psi_2) = \int d^3r\, \Psi_1^\dagger(\mathbf{r})\Psi_2(\mathbf{r}). \tag{3.15}$$

Wave functions of physical photons are built from positive-energy solutions of the eigenvalue problem for the Hamiltonian,

$$\hat{H}\Psi(\mathbf{r}) = E\Psi(\mathbf{r}). \tag{3.16}$$

There is simple relation between positive-energy solutions and negative-energy solutions. One may obtain all solutions for the negative energies just by an interchange of upper and lower components since $\rho_1\hat{H}\rho_1 = -\hat{H}$. Such a simple

symmetry of solutions is a result of photons being identical with antiphotons and it is not found, in general, for particles whose antiparticles are physically distinct. For example, the solutions of the wave functions describing electrons in the Coulomb potential of the proton are quite different from the wave function of positrons moving in the same potential. In the first case the potential is attractive (bound states), while in the second case it is repulsive (only scattering states).

Explicit solutions of the energy eigenvalue problem for photons are easily obtained in free space but in the presence of a medium this can be done only in special cases. In this respect, wave mechanics of photons is not much different from wave mechanics of massive particles, where explicit solutions can also be found only for special potentials.

§4.   Photon wave function in momentum representation

The most thorough textbook treatment of quantum mechanics of photons has been given by Akhiezer and Berestetskii [1965] who devoted the whole long chapter to this problem. Their discussion is limited to momentum representation except for a brief subsection under a characteristic title: "Impossibility of introducing a photon wave function in the coordinate representation". This impossibility will be addressed in §8.

Wave mechanics of photons in momentum representation can be derived directly from relativistic quantum kinematics and group representation theory but here the analysis will be based on the Fourier representation of the photon wave function in coordinate representation.

In this review I shall use traditionally the wave vector $\mathbf{k}$ instead of the photon momentum vector $\mathbf{p} = \hbar\mathbf{k}$ as the argument of the wave function in momentum space. The explicit introduction of Planck's constant is necessary only when the proper normalization of the wave function is needed.

4.1.   PHOTON WAVE FUNCTION AS A FOURIER INTEGRAL

The standard procedure for solving the wave equations (1.5) or (2.3) is based on Fourier transformation. Since every solution of eqs. (1.5) and (1.6) is a solution of the d'Alembert equation

$$(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta)\mathbf{F}_\pm(\mathbf{r}, t) = 0, \tag{4.1}$$

the vectors $\mathbf{F}_\pm$ can be represented as superpositions of plane waves

$$\mathbf{F}_+(\mathbf{r},t) = \sqrt{\hbar c} \int \frac{d^3k}{(2\pi)^3} \left[\mathbf{f}_+(\mathbf{k})\, e^{-i\omega t + i\mathbf{k}\cdot\mathbf{r}} + \mathbf{f}_-^*(\mathbf{k})\, e^{i\omega t - i\mathbf{k}\cdot\mathbf{r}} \right], \tag{4.2}$$

$$\mathbf{F}_-(\mathbf{r},t) = \sqrt{\hbar c} \int \frac{d^3k}{(2\pi)^3} \left[\mathbf{f}_-(\mathbf{k})\, e^{-i\omega t + i\mathbf{k}\cdot\mathbf{r}} + \mathbf{f}_+^*(\mathbf{k})\, e^{i\omega t - i\mathbf{k}\cdot\mathbf{r}} \right], \tag{4.3}$$

where $\omega = c|\mathbf{k}|$ and the factor $\sqrt{\hbar c}$ has been introduced for future convenience. The remaining integral has the dimension of $1/\text{length}^2$, so that the Fourier coefficients $\mathbf{f}_\pm$ have the dimension of length. It has already been taken into account in (4.2) and (4.3) that the vectors $\mathbf{F}_+$ and $\mathbf{F}_-$ are complex conjugate of each other. In order to fulfill Maxwell's equations, the two complex vectors $\mathbf{f}_+(\mathbf{k})$ and $\mathbf{f}_-(\mathbf{k})$ must satisfy the set of linear, algebraic equations that result from (1.5) and (1.6), respectively,

$$i\mathbf{k} \times \mathbf{f}_\pm(\mathbf{k}) \ = \ \pm|\mathbf{k}|\mathbf{F}_\pm(\mathbf{k}), \tag{4.4}$$
$$\mathbf{k}\cdot\mathbf{f}_\pm(\mathbf{k}) \ = \ 0. \tag{4.5}$$

Actually, the second equation is superfluous since it follows from the first. Solutions of these equations are determined up to a complex factor. Denoting by $\mathbf{e}(\mathbf{k})$ a normalized solution of the first equation taken with a plus sign

$$i\mathbf{k} \times \mathbf{e}(\mathbf{k}) \ = \ |\mathbf{k}|\mathbf{e}(\mathbf{k}), \tag{4.6}$$
$$\mathbf{e}^*(\mathbf{k})\cdot\mathbf{e}(\mathbf{k}) \ = \ 1, \tag{4.7}$$

one can express the vectors $\mathbf{f}_\pm(\mathbf{k})$ in the form

$$\mathbf{f}_+(\mathbf{k}) = \mathbf{e}(\mathbf{k})f(\mathbf{k},1), \quad \mathbf{f}_-(\mathbf{k}) = \mathbf{e}^*(\mathbf{k})f(\mathbf{k},-1). \tag{4.8}$$

The two complex functions $f(\mathbf{k},\lambda)$, where $\lambda = \pm 1$, describe the independent degrees of freedom of the free electromagnetic field. The vector $\mathbf{e}(\mathbf{k})$ can be decomposed into two real vectors $\mathbf{l}_i(\mathbf{k})$ that form together with the unit vector $\mathbf{n}(\mathbf{k}) = \mathbf{k}/|\mathbf{k}|$ an orthonormal set

$$\mathbf{e}(\mathbf{k}) = (\mathbf{l}_1(\mathbf{k}) + i\mathbf{l}_2(\mathbf{k}))/\sqrt{2}, \quad \mathbf{l}_i(\mathbf{k})\cdot\mathbf{l}_j(\mathbf{k}) = \delta_{ij}, \tag{4.9}$$
$$\mathbf{n}(\mathbf{k})\cdot\mathbf{l}_i(\mathbf{k}) = 0, \quad \mathbf{l}_1(\mathbf{k}) \times \mathbf{l}_2(\mathbf{k}) = \mathbf{n}(\mathbf{k}). \tag{4.10}$$

The only freedom left in the definition of $\mathbf{e}(\mathbf{k})$ is its phase. A multiplication by a phase factor amounts to a rotation of the vectors $\mathbf{l}_i(\mathbf{k})$ around the vector $\mathbf{n}(\mathbf{k})$. The same freedom characterizes the coefficient functions $f(\mathbf{k},\lambda)$. This

phase may, in general, depend on $\mathbf{k}$ and it plays an important role in the study of the photon wave function in momentum representation. The final form of the Fourier representation for vectors $\mathbf{F}_\pm$ is

$$\mathbf{F}_+(\mathbf{r},t) = \sqrt{\hbar c}\int \frac{d^3k}{(2\pi)^3}\mathbf{e}(\mathbf{k})\big[f(\mathbf{k},1)\,e^{-i\omega t + i\mathbf{k}\cdot\mathbf{r}} + f^*(\mathbf{k},-1)\,e^{i\omega t - i\mathbf{k}\cdot\mathbf{r}}\big], \quad (4.11)$$

$$\mathbf{F}_-(\mathbf{r},t) = \sqrt{\hbar c}\int \frac{d^3k}{(2\pi)^3}\mathbf{e}^*(\mathbf{k})\big[f(\mathbf{k},-1)\,e^{-i\omega t + i\mathbf{k}\cdot\mathbf{r}} + f^*(\mathbf{k},1)\,e^{i\omega t - i\mathbf{k}\cdot\mathbf{r}}\big]. \quad (4.12)$$

4.2.   INTERPRETATION OF FOURIER COEFFICIENTS

In free space, the energy, momentum, angular momentum, and moment of energy of the classical electromagnetic field are given by the expressions (1.8)–(1.11). With the help of the formulas (4.11) of (4.12) they can be expressed in terms of the coefficient functions $f(\mathbf{k},\lambda)$ (cf., for example, Bialynicki-Birula and Bialynicka-Birula [1975])

$$E = \sum_\lambda \int \frac{d^3k}{(2\pi)^3|\mathbf{k}|}\hbar\omega f^*(\mathbf{k},\lambda)f(\mathbf{k},\lambda), \qquad\qquad (4.13)$$

$$\mathbf{P} = \sum_\lambda \int \frac{d^3k}{(2\pi)^3|\mathbf{k}|}\hbar\mathbf{k}f^*(\mathbf{k},\lambda)f(\mathbf{k},\lambda), \qquad\qquad (4.14)$$

$$\mathbf{M} = \sum_\lambda \int \frac{d^3k}{(2\pi)^3|\mathbf{k}|}f^*(\mathbf{k},\lambda)(\hbar\mathbf{k}\times\frac{1}{i}\mathbf{D_k} + \lambda\hbar\frac{\mathbf{k}}{|\mathbf{k}|})f(\mathbf{k},\lambda), \qquad (4.15)$$

$$\mathbf{N} = \sum_\lambda \int \frac{d^3k}{(2\pi)^3|\mathbf{k}|}f^*(\mathbf{k},\lambda)i\hbar\omega\mathbf{D_k}f(\mathbf{k},\lambda), \qquad\qquad (4.16)$$

where

$$\mathbf{D_k} = \partial_\mathbf{k} + i\lambda\boldsymbol{\alpha}(\mathbf{k}), \qquad\qquad (4.17)$$

and

$$\boldsymbol{\alpha}(\mathbf{k}) = \mathbf{l}_1(\mathbf{k})\cdot\partial_\mathbf{k}\mathbf{l}_2(\mathbf{k}) - \mathbf{l}_2(\mathbf{k})\cdot\partial_\mathbf{k}\mathbf{l}_1(\mathbf{k}). \qquad\qquad (4.18)$$

The operation $\mathbf{D_k}$ is a natural covariant derivative on the light cone (Staruszkiewicz [1973], Bialynicki-Birula and Bialynicka-Birula [1975], Bialynicki-Birula and Bialynicka-Birula [1987]). Note that this operation depends through the vector $\boldsymbol{\alpha}(\mathbf{k})$ on the phase convention for the polarization vector $\mathbf{e}(\mathbf{k})$. It is interesting to note that the vector $\boldsymbol{\alpha}(\mathbf{k})$ has similar properties to the electromagnetic vector potential. It changes

by a gradient under a change of the phase, but its curl is uniquely defined. Indeed, it follows from the definition (4.18) of $\boldsymbol{\alpha}(\mathbf{k})$ and from the orthonormality conditions (4.10) that the vector $\boldsymbol{\alpha}(\mathbf{k})$ obeys the equation

$$\partial_i \alpha_j - \partial_j \alpha_i = -\epsilon_{ijk} n_k / \mathbf{k}^2. \tag{4.19}$$

This equation determines the Berry phase (Bialynicki-Birula and Bialynicka-Birula [1987]) in the propagation of photons.

The formulas (4.13) and (4.14) indicate that $f(\mathbf{k}, \pm 1)$ describe field amplitudes with energy $\hbar\omega$ and momentum $\hbar\mathbf{k}$. The formula (4.15) shows that $f(\mathbf{k}, 1)$ and $f(\mathbf{k}, -1)$ describe field amplitudes with positive and negative helicity since their contribution to the component of angular momentum in the direction of momentum is equal to $\pm\hbar$, respectively.

The functions $f(\mathbf{k}, \lambda)$ have actually a dual interpretation. In classical theory they yield full information about the electromagnetic field. In wave mechanics of the photon, these functions are the components of the photon wave function in momentum representation. In order to distinguish these two cases, I shall denote the two components of the photon wave function in momentum representation by a new symbol $\phi(\mathbf{k}, \lambda)$. Wave function must be normalized and the proper normalization of the photon wave function $\phi(\mathbf{k}, \lambda)$ is discussed in §5.

The expansion of the photon wave function into plane waves has the following form

$$\Psi(\mathbf{r}, t) = \sqrt{\hbar c} \int \frac{d^3 k}{(2\pi)^3} \begin{bmatrix} \mathbf{e}(\mathbf{k}, 1) \phi(\mathbf{k}, 1) \\ \mathbf{e}(\mathbf{k}, -1) \phi(\mathbf{k}, -1) \end{bmatrix} e^{-i\omega t + i\mathbf{k} \cdot \mathbf{r}}, \tag{4.20}$$

where

$$\mathbf{e}(\mathbf{k}, 1) = \mathbf{e}(\mathbf{k}), \quad \mathbf{e}(\mathbf{k}, -1) = \mathbf{e}^*(\mathbf{k}). \tag{4.21}$$

The integral (4.20) defines a certain (continuous) superposition of the wave functions $\phi(\mathbf{k}, \lambda)$ with different values of the wave vector. If the photon wave function in momentum representation is accepted as a legitimate concept, then the superpositions of such functions must also be accepted. Those who are not sure about the meaning of a continuous superpositions, in the form of an integral, may restrict the electromagnetic field to a box and replace the integral by a discrete sum.

4.3. TRANSFORMATION PROPERTIES OF THE PHOTON WAVE FUNCTION IN MOMENTUM REPRESENTATION

Transformation rules for the wave function in momentum representation may be derived from the transformation properties of the Riemann-Silberstein vector.

They are the same in the classical theory of the electromagnetic field and in the quantum theory of photons. Under space and time translations, functions $\phi(\mathbf{k}, \lambda)$ are multiplied by the phase factors

$$\phi'(\mathbf{k}, \lambda) = \exp(-i\omega t_0 + i\mathbf{k}\cdot\mathbf{r}_0)\phi(\mathbf{k}, \lambda), \tag{4.22}$$

where $(\mathbf{r}_0, t_0)$ is the four-vector of translation. From expressions (4.13) and (4.14) one may deduce that under rotations and Lorentz transformations, the functions $\phi(\mathbf{k}, \lambda)$ are also multiplied by phase factors

$$\phi'(\mathbf{k}', \lambda) = \exp(-i\lambda\Theta(\mathbf{k}, \Lambda))\phi(\mathbf{k}, \lambda), \tag{4.23}$$

where the phase function $\Theta(\mathbf{k}, \Lambda)$ depends on the Poincaré transformation Λ. This transformation property can easily be derived from the transformation law of the energy-momentum four-vector for the electromagnetic field. The right hand side in the formulas (4.13) and (4.14) has three factors: the integration volume $d^3k/(2\pi)^3|\mathbf{k}|$, the four-vector $(\mathbf{k}, \omega)$, and the moduli squared of $\phi(\mathbf{k}, \lambda)$. The integration volume is an invariant (cf., for example, Weinberg [1995], p. 67) and therefore $|\phi(\mathbf{k}, \lambda)|^2$ must also be invariant. An explicit form of the phase $\Theta(\mathbf{k}, \Lambda)$ corresponding to a given Poincaré transformation Λ can be given (cf., for example, Amrein [1969]) but it is not very illuminating.

§5.   Probabilistic interpretation

Probabilistic interpretation of wave mechanics requires, first of all, a definition of the scalar product $\langle\Psi_1|\Psi_2\rangle$ that is to be used in the calculation of transition probabilities. The modulus squared of the scalar product of two normalized wave functions $|\langle\Psi_1|\Psi_2\rangle|^2$ determines the probability of finding a photon in the state Ψ_1 when the photon is in the state Ψ_2. The probability, of course, must be a pure number and — as a true observable — must be invariant under all Poincaré transformations. The most obvious definition of the scalar product (3.15) can not be used because it is neither Poincaré invariant nor dimensionally correct. There is essentially only one candidate for the correct scalar product. Its heuristic derivation is the easiest in momentum representation.

5.1.   SCALAR PRODUCT

According to quantum mechanics each photon with momentum $\hbar\mathbf{k}$ carries energy $\hbar\omega$. Thus, the total number of photons N present in the electromagnetic

field is obtained by dividing the integrand in the formula (4.13) by $\hbar\omega$,

$$N = \sum_\lambda \int \frac{d^3k}{(2\pi)^3|\mathbf{k}|}|f(\mathbf{k},\lambda)|^2. \tag{5.1}$$

A photon wave function describes just one photon. The normalized wave function must, therefore, satisfy the condition $N = 1$. Normalized photon wave functions in momentum representation satisfy the normalization condition

$$\sum_\lambda \int \frac{d^3k}{(2\pi)^3|\mathbf{k}|}|\phi(\mathbf{k},\lambda)|^2 = 1. \tag{5.2}$$

The form of the scalar product can be deduced from the expression for the norm and it reads

$$\langle\Psi_1|\Psi_2\rangle = \sum_\lambda \int \frac{d^3k}{(2\pi)^3|\mathbf{k}|}\phi_1^*(\mathbf{k},\lambda)\phi_2(\mathbf{k},\lambda). \tag{5.3}$$

This scalar product can be also expressed in terms of the photon wave functions in coordinate representation by inverting the Fourier transformation in eq. (4.20)

$$\phi(\mathbf{k},\lambda) = \frac{1}{\sqrt{\hbar c}}\mathbf{e}^*(\mathbf{k},\lambda)\cdot\int d^3r \exp(-i\mathbf{k}\cdot\mathbf{r})\Psi(\mathbf{r},t), \tag{5.4}$$

where the scalar product with $\mathbf{e}^*(\mathbf{k},\lambda)$ is evaluated for upper and lower components separately and for each λ only one of them does not vanish. Upon substituting this expression into (5.3), interchanging the order of integrations, and using the following properties of the vectors $\mathbf{e}(\mathbf{k},\lambda)$

$$\mathbf{e}^*(\mathbf{k},\lambda)\cdot\mathbf{e}(\mathbf{k},\lambda') = \delta_{\lambda,\lambda'}, \tag{5.5}$$

$$\sum_\lambda e_i^*(\mathbf{k},\lambda)e_j(\mathbf{k},\lambda) = \delta_{ij} - n_i(\mathbf{k})n_j(\mathbf{k}), \tag{5.6}$$

one obtains the following expression for the scalar product in coordinate representation

$$\langle\Psi_1|\Psi_2\rangle = \frac{1}{2\pi^2\hbar c}\int d^3r \int d^3r' \Psi_1^\dagger(\mathbf{r})\frac{1}{|\mathbf{r}-\mathbf{r}'|^2}\Psi_2(\mathbf{r}'). \tag{5.7}$$

The norm associated with this scalar product is

$$N = \|\Psi\|^2 = \frac{1}{2\pi^2\hbar c}\int d^3r \int d^3r' \Psi^\dagger(\mathbf{r})\frac{1}{|\mathbf{r}-\mathbf{r}'|^2}\Psi(\mathbf{r}'). \tag{5.8}$$

The scalar product (5.7) and the associated norm (5.8) for photon wave functions have been arrived at by numerous authors starting from various premises. Gross [1964] has proven that this scalar product and this norm are invariant not only under Poincaré transformations but also under conformal transformations. Zeldovich [1965] derived the formula (5.8) for the number of photons in terms of the electromagnetic field vectors. Recently, the norm (5.8) has been found very useful in the formulation of wavelet electrodynamics (Kaiser [1992]). The same expression (5.3) for the scalar product can also be derived by considering quantum-mechanical expectation values. That approach has been used by Good [1957] and is presented below.

5.2. EXPECTATION VALUES OF PHYSICAL QUANTITIES

In wave mechanics of photons, the normalized photon wave function $\phi(\mathbf{k}, \lambda)$ replaces the classical field amplitudes $f(\mathbf{k}, \lambda)$. The classical expressions for the energy, momentum, angular momentum, and moment of energy become the formulas for the quantum-mechanical expectation values

$$\langle E \rangle = \langle \Psi | \hat{H} | \Psi \rangle, \quad \langle \mathbf{P} \rangle = \langle \Psi | \hat{\mathbf{P}} | \Psi \rangle, \tag{5.9}$$

$$\langle \mathbf{M} \rangle = \langle \Psi | \hat{\mathbf{J}} | \Psi \rangle, \quad \langle \mathbf{N} \rangle = \langle \Psi | \hat{\mathbf{K}} | \Psi \rangle. \tag{5.10}$$

These equations compared with the formulas (4.13)–(4.16) enable one to identify the operators $\hat{H}$, $\hat{\mathbf{P}}$, $\hat{\mathbf{J}}$, and $\hat{\mathbf{K}}$ in momentum representation as

$$\hat{H} = \hbar\omega, \tag{5.11}$$

$$\hat{\mathbf{P}} = \hbar\mathbf{k}, \tag{5.12}$$

$$\hat{\mathbf{J}} = \mathbf{k} \times \frac{\hbar}{i}\mathbf{D_k} + \lambda\hbar\frac{\mathbf{k}}{|\mathbf{k}|}, \tag{5.13}$$

$$\hat{\mathbf{K}} = \hbar\omega\frac{\hbar}{i}\mathbf{D_k}. \tag{5.14}$$

The operators $\hat{H}$, $\hat{\mathbf{P}}$, $\hat{\mathbf{J}}$, and $\hat{\mathbf{K}}$ are Hermitian with respect to the scalar product given by the formula (5.3).

The formulas (5.11)–(5.14) are fully consistent with the interpretation of $\phi(\mathbf{k}, \lambda)$ as the probability amplitude in momentum representation. The probability density to find the photon with the momentum $\hbar\mathbf{k}$ and the helicity λ is

$$\text{Probability density} = \frac{|\phi(\mathbf{k}, \lambda)|^2}{(2\pi)^3|\mathbf{k}|}. \tag{5.15}$$

In order to express the quantum-mechanical expectation values (5.11)–(5.14) in coordinate representation one must identify the proper form of the scalar product for the photon wave function Ψ. This identification has already been made by Good [1957] who compared the classical formula for the energy of the electromagnetic field with the quantum-mechanical expression involving the Hamiltonian

$$\langle E \rangle = \langle \Psi | \hat{H} | \Psi \rangle, \tag{5.16}$$

and came to the conclusion that the scalar product for the photon wave function has to be modified as follows

$$\langle \Psi_1 | \Psi_2 \rangle = \int d^3 r \, \Psi_1^\dagger \frac{1}{\hat{H}} \Psi_2. \tag{5.17}$$

It is assumed here that the wave functions are built from positive energy states only and that guarantees the positive definiteness of the norm

$$\|\Psi\|^2 = \int d^3 r \, \Psi^\dagger \frac{1}{\hat{H}} \Psi, \tag{5.18}$$

associated with that scalar product. This form of the scalar product leads to the following expectation values of the energy, momentum, angular momentum, and moment of energy operators

$$\langle E \rangle = \int d^3 r \, \Psi^\dagger \hat{H}^{-1} \left(\mathbf{s} \cdot \frac{\hbar}{i} \nabla \right) \Psi, \tag{5.19}$$

$$\langle \mathbf{P} \rangle = \int d^3 r \, \Psi^\dagger \hat{H}^{-1} \frac{\hbar}{i} \nabla \Psi, \tag{5.20}$$

$$\langle \mathbf{M} \rangle = \int d^3 r \, \Psi^\dagger \hat{H}^{-1} \left(\mathbf{r} \times \frac{\hbar}{i} \nabla + \hbar \mathbf{s} \right) \Psi, \tag{5.21}$$

$$\langle \mathbf{N} \rangle = \int d^3 r \, \Psi^\dagger \hat{H}^{-1} \hat{H} \mathbf{r} \Psi. \tag{5.22}$$

Thus, the operators $\hat{H}$, $\hat{\mathbf{P}}$, $\hat{\mathbf{J}}$, and $\hat{\mathbf{K}}$ in coordinate representation have the form

$$\hat{H} = c \left(\mathbf{s} \cdot \frac{\hbar}{i} \nabla \right), \tag{5.23}$$

$$\hat{\mathbf{P}} = \frac{\hbar}{i} \nabla, \tag{5.24}$$

$$\hat{\mathbf{J}} = \mathbf{r} \times \frac{\hbar}{i} \nabla + \hbar \mathbf{s}, \tag{5.25}$$

$$\hat{\mathbf{K}} = \hat{H} \mathbf{r}. \tag{5.26}$$

All these operators preserve the divergence condition (1.6) and they are Hermitian with respect to the scalar product (5.17). It is also reassuring to note that the quantum-mechanical operators of momentum and angular momentum in coordinate representation have the same form as in standard quantum mechanics. This can be taken as another indication that $\Psi(\mathbf{r}, t)$ is a legitimate and useful object.

One may prove directly (without using the Fourier expansions) with the help of the following identities

$$\hat{H}\rho_3\mathbf{s}/c\Psi = \frac{\hbar}{i}\nabla\Psi, \quad \hat{H}\rho_3\hat{\mathbf{r}} \times \mathbf{s}/c\Psi = (\hat{\mathbf{r}} \times \frac{\hbar}{i}\nabla + \hbar\mathbf{s})\Psi, \tag{5.27}$$

and with the use of eq. (2.4) that the expectation values (5.19)–(5.21) reduce to the classical expressions (1.8)–(1.11) when the wave function is replaced by the classical electromagnetic field.

The scalar product (5.17) in coordinate representation has been obtained from the scalar product (5.3) in momentum representation. However, the scalar product that contains the division by the Hamiltonian can be derived on more general grounds and its definition does not depend on the choice of representation. It has been shown (Segal [1963], Ashtekar and Magnon [1975]) that such a scalar product is a general feature of geometric quantization in field theory.

Even though the number of photons is given by a double integral, so that there is no local expression for the photon probability density in coordinate space, the expression for the energy has the form of a single integral over $|\Psi(\mathbf{r})|^2$. Therefore, one may introduce a tentative notion of the "average photon energy in a region of space" and try to associate a probabilistic interpretation of the photon wave function with this quantity (Bialynicki-Birula [1994], Sipe [1995]). More precisely, the quantity $p_E(\Omega)$,

$$p_E(\Omega) = \frac{\int_\Omega d^3r\Psi^\dagger(\mathbf{r})\Psi(\mathbf{r})}{\langle E\rangle}, \tag{5.28}$$

may be interpreted as the probability to find the energy of the photon localized in the region Ω. In other words, $p_E(\Omega)$ is the fraction of the average total energy of the photon associated with the region Ω. The probability density $\rho_E(\mathbf{r}, t)$ to find the energy of the photon at the point $\mathbf{r}$,

$$\rho_E(\mathbf{r}, t) = \frac{\Psi^\dagger(\mathbf{r}, t)\Psi(\mathbf{r}, t)}{\langle E\rangle}, \tag{5.29}$$

is properly normalized to one and it also satisfies the continuity equation

$$\partial_t\rho_E(\mathbf{r}, t) + \nabla\cdot\mathbf{j}_E(\mathbf{r}, t) = 0, \tag{5.30}$$

with the normalized average energy flux

$$\mathbf{j}_E(\mathbf{r}, t) = \frac{\Psi^\dagger(\mathbf{r}, t)\rho_3 \mathbf{s}\Psi(\mathbf{r}, t)}{\langle E \rangle}, \tag{5.31}$$

as the probability current. The direct connection between the wave function $\Psi(\mathbf{r}, t)$ and the average energy density justifies the name "the energy wave function" used by Mandel and Wolf [1995]. It is understandable that the localization of photons is associated with their energy because photons do not carry other attributes like charge, fermion number, or rest mass. It is worth noting that for gravitons not only the probability but even the energy can not be localized (cf., for example, Weinberg and Witten [1980]). The probabilistic interpretation of the energy wave function Ψ is still subject to all the limitations arising from the lack of the photon position operator, as discussed in §8. In particular, there are no projection operators whose expectation values would give the probabilities $p_E(\Omega)$.

The transition amplitudes, the operators representing important physical quantities, and the expectation values can be expressed with equal ease in momentum representation and in coordinate representation. For photons moving in empty space both representations are completely equivalent and give the same results. The only relevant issue is whether a particular superposition of wave functions in momentum representation is useful for the description of quantum states of the photon. The distinguished and unique feature of the superposition given by the Fourier integrals (4.20) is that they represent *local* fields. They have local transformation properties (3.10) and they satisfy local boundary conditions. Therefore, for photons moving in an inhomogeneous or bounded medium, it is the coordinate representation that is preferred because only in this representation one may easily take into account the properties of the medium (cf. §6).

5.3.   CONNECTION WITH LANDAU-PEIERLS WAVE FUNCTION

One may easily convert the scalar product (5.7) into a standard form containing a single integration with the use of the following identity

$$\frac{1}{16\pi}\int d^3r \frac{1}{|\mathbf{r}-\mathbf{r}'|^{5/2}}\frac{1}{|\mathbf{r}-\mathbf{r}''|^{5/2}} = \frac{1}{|\mathbf{r}'-\mathbf{r}''|^2}. \tag{5.32}$$

This enables one to convert the double integral (5.7) into a single integral

$$\langle \Psi_1 | \Psi_2 \rangle = \int d^3r \Phi_1^\dagger(\mathbf{r})\Phi_2(\mathbf{r}). \tag{5.33}$$

The new functions Φ are the Landau-Peierls wave functions and they are related to the photon wave functions Ψ through the formula

$$\Phi(\mathbf{r}) = \frac{\pi}{\sqrt{\hbar c}} \int d^3 r' \frac{1}{(2\pi|\mathbf{r} - \mathbf{r}'|)^{5/2}} \Psi(\mathbf{r}'). \tag{5.34}$$

The form of the scalar product for the Landau-Peierls wave functions is simple but one must pay for this simplicity with the nonlocality of the wave functions. There is also a simple mathematical argument that shows shortcomings of the Landau-Peierls wave function. While for every integrable wave function Ψ the transformation (5.34) defines the Landau-Peierls wave function Φ, the inverse transformation is singular since it contains a nonintegrable kernel $|\mathbf{r} - \mathbf{r}'|^{7/2}$ (Amrein [1969], Mandel and Wolf [1995]). This leads to a paradox that for many "reasonable" functions Φ (for example, for every function that becomes zero abruptly at the boundary) the energy density is infinite. Thus, it is much more natural to treat Ψ as the primary and Φ as the derived object.

§6.   Eigenvalue problems for the photon wave function

In wave mechanics of photons as in wave mechanics of massive particles, one may study eigenvalues and eigenfunctions of various interesting observables. The most important observables, of course, are the momentum, angular momentum, energy, and moment of energy — the generators of the Poincaré group. The eigenfunctions of these observables will be given in coordinate representation to underscore the validity and usefulness of the photon wave function in this representation.

6.1.   EIGENVALUE PROBLEMS FOR MOMENTUM AND ANGULAR MOMENTUM

The eigenvalue problems for the components of the photon momentum operator have the standard quantum-mechanical form

$$\hat{P}_i \ \Psi(\mathbf{r}) = \hbar k_i \ \Psi(\mathbf{r}), \tag{6.1}$$

and its solutions depend on $\mathbf{r}$ through the exponential functions $\exp(i\mathbf{k}\cdot\mathbf{r})$.

The eigenvalue problem for the photon angular momentum also has the standard quantum-mechanical form. It contains, as usual, the eigenvalue problem for the z-component of the total angular momentum

$$\hat{J}_z \ \Psi(\mathbf{r}) = \hbar M \Psi(\mathbf{r}), \tag{6.2}$$

and the eigenvalue problem for the square of the total angular momentum

$$\hat{\mathbf{J}}^2 \, \Psi(\mathbf{r}) = \hbar^2 J(J+1) \, \Psi(\mathbf{r}). \tag{6.3}$$

The solutions of eqs. (6.2) and (6.3) are well known vector spherical harmonics (cf., for example, Messiah [1961]). The direct connection between the quantum-mechanical eigenvalue problems and multipole expansion in classical electromagnetism has been explored systematically for the first time by Molière [1949].

6.2. EIGENVALUE PROBLEM FOR THE MOMENT OF ENERGY

The solution of the eigenvalue problem for the moment of energy shows the versatility of the calculational methods based on the coordinate representation and sheds some light on the problem of the localizability of the photon that is discussed in §8. Of course, the same result can be obtained by Fourier transforming the solution of the eigenvalue problem obtained in momentum representation.

The three components of the moment of energy, like the components of angular momentum, do not commute among themselves. Therefore, the eigenvalue problem can be posed only for one component at a time. Choosing, for definiteness, the z-component, one obtains the following eigenvalue equation

$$- i\rho_3 \big(\mathbf{s}\!\cdot\!\nabla\big) z \, \Psi = \kappa \, \Psi. \tag{6.4}$$

The solution of this equation becomes unique when one chooses two additional eigenvalue equations to be solved concurrently. For example, eq. (6.4) can be solved together with the eigenvalue problems for the x and y components of momentum since the three operators $\hat{P}_x$, $\hat{P}_y$, and $\hat{K}_z$ commute. The solutions for the upper and lower components of the wave function differ only in the sign of κ and one can solve them independently. When the wave function in the form $\Psi = \exp(ik_x x + ik_y y)(\psi_x(z), \psi_y(z), \psi_z(z))$ is substituted into (6.4), one obtains a set of ordinary differential equations

$$ik_y z\psi_z - (z\psi_y)' = \kappa \, \psi_x, \tag{6.5}$$

$$(z\psi_x)' - ik_x z\psi_z = \kappa \, \psi_y, \tag{6.6}$$

$$ik_x z\psi_y - ik_y z\psi_x = \kappa \, \psi_z, \tag{6.7}$$

where the prime denotes the differentiation with respect to z. These equations are solved by the following substitution ($k_\perp^2 = k_x^2 + k_y^2$)

$$\psi_x = \frac{i}{k_\perp^2 z}(k_y \kappa + k_x \frac{d}{dz})\psi_z, \tag{6.8}$$

$$\psi_y = \frac{i}{k_\perp^2 z}(-k_x \kappa + k_y \frac{d}{dz})\psi_z, \tag{6.9}$$

which results in a Bessel-type equation for ψ_z,

$$z^2 \psi_z'' + z \psi_z' + (\kappa^2 - k_\perp^2 z^2) \psi_z = 0. \tag{6.10}$$

The physically acceptable solution of this equation is given by the Macdonald function of the imaginary index

$$\psi_z(z) = K_{i\kappa}(k_\perp z) = \int_0^\infty dt \, e^{-k_\perp z \cosh t} \cos(\kappa t). \tag{6.11}$$

The other solution grows exponentially when $z \to \infty$ and must be rejected. The physical solution falls off exponentially for large $|z|$ and represents a photon state that is localized as much as possible in the z-direction. The remaining two components of the eigenfunction are obtained from eqs. (6.8) and (6.9). The photon wave functions that describe eigenstates of $\hat{K}_z$ are not normalizable, because the spectrum of the eigenvalues is continuous: κ can be any real number.

6.3. PHOTON PROPAGATION ALONG AN OPTICAL FIBER AS A QUANTUM MECHANICAL BOUND STATE PROBLEM

The eigenvalue problem for the photon energy operator in the absence of a medium is solved by Fourier transformation as described in §4. In the presence of a medium, one can search for eigenstates of the photon Hamiltonian closely following the path traveled in nonrelativistic wave mechanics of massive particles. This procedure usually involves selecting a set of operators commuting with the Hamiltonian and then solving the appropriate set of eigenvalue equations. The photon propagation along an infinite cylindrical optical fiber (cf., for example, Bialynicki-Birula [1994]) is a good illustration of this approach. In order to take care of the boundary conditions at the surface of the fiber, one must work in the coordinate representation.

Consider an infinite, cylindrical optical fiber of diameter a characterized by a dielectric permittivity ϵ. The symmetry of the problem suggests the inclusion in the set of commuting operators, in addition to the Hamiltonian, the projections of the momentum operator and the total angular momentum on the direction of the fiber axis. In cylindrical coordinates the eigenvalue equations for the z-components of momentum and angular momentum and the Hamiltonian have the form

$$-i\partial_z \, \Psi \;=\; k_z \Psi, \tag{6.12}$$

$$\left(-i\partial_\varphi + (\mathbf{s} \cdot \mathbf{e}_z)\right) \Psi \;=\; M\Psi, \tag{6.13}$$

$$-i\rho_3\left(\mathbf{s} \cdot \left(\mathbf{e}_\rho \partial_\rho + \frac{1}{\rho}\mathbf{e}_\varphi \partial_\varphi + \mathbf{e}_z \partial_z\right)\right) \Psi \;=\; \frac{\omega}{v} \, \Psi, \tag{6.14}$$

where $\mathbf{e}_\rho$, $\mathbf{e}_\varphi$, and $\mathbf{e}_z$ are the unit vectors along the coordinate lines and v equals to c outside the fiber. Due to the symmetry of the problem, there is no coupling between the upper and lower components of Ψ and the solution of these eigenvalue equations can be sought in the form of a three dimensional vector,

$$\Psi = \mathbf{e}_\rho \psi_\rho + \mathbf{e}_\varphi \psi_\varphi + \mathbf{e}_z \psi_z. \tag{6.15}$$

In order to separate the variables and obtain a set of ordinary differential equations, one needs the following differential and algebraic relations

$$\partial_\varphi \mathbf{e}_\rho = \mathbf{e}_\varphi, \quad \partial_\varphi \mathbf{e}_\varphi = -\mathbf{e}_\rho, \tag{6.16}$$

$$(\mathbf{s} \cdot \mathbf{e}_\rho)\mathbf{e}_\varphi = i\mathbf{e}_z, \quad (\mathbf{s} \cdot \mathbf{e}_z)\mathbf{e}_\rho = i\mathbf{e}_\varphi, \quad (\mathbf{s} \cdot \mathbf{e}_\varphi)\mathbf{e}_z = i\mathbf{e}_\rho. \tag{6.17}$$

All unlisted terms of the type (6.16) and (6.17) vanish. The dependence on φ and z of all three components ψ_ρ, ψ_φ, and ψ_z of the photon wave function can be separated out on the basis of eqs. (6.12) and (6.13),

$$\psi = \exp(ik_z z)\exp(iM\varphi)f(\rho). \tag{6.18}$$

The three ρ-dependent components of the wave function satisfy the equations

$$-\frac{M}{\rho}f_z + k_z f_\varphi = \frac{\omega}{v}(if_\rho), \tag{6.19}$$

$$-\partial_\rho f_z + k_z(if_\rho) = \frac{\omega}{v}f_\varphi, \tag{6.20}$$

$$\frac{1}{\rho}\partial_\rho \rho f_\varphi - \frac{M}{\rho}(if_\rho) = \frac{\omega}{v}f_z. \tag{6.21}$$

These equations lead to a Bessel equation for f_z

$$\left[\partial_\rho^2 + \frac{1}{\rho}\partial_\rho - \frac{m^2}{\rho^2} + k_\perp^2\right]f_z = 0, \tag{6.22}$$

where $k_\perp^2 = \omega^2/v^2 - k_z^2$. The remaining two functions f can be determined in terms of f_z,

$$f_\rho = ik_\perp^{-2}\left(\frac{\omega M}{v\rho} + k_z \partial_\rho\right)f_z, \tag{6.23}$$

$$f_\varphi = k_\perp^{-2}\left(\frac{Mk_z}{\rho} + \frac{\omega}{v}\partial_\rho\right)f_z. \tag{6.24}$$

The photon wave function obeys the Bessel equation inside the fiber with one value of $k_\perp$ and with a different values of $k_\perp$ in the surrounding free space. The behavior of the solution of eq. (6.22) depends on whether $k_\perp$ is real or imaginary. A general solution of this equation is either (for real $k_\perp$) a linear combination of Bessel functions of the first kind $J_M(\rho)$ and the second kind $Y_M(\rho)$ or (for imaginary $k_\perp$) a linear combination of modified Bessel functions $I_M(\rho)$ and $K_M(\rho)$. In full analogy with the problem of a potential well in quantum mechanics, one can search for bound states in the transverse direction by matching a regular oscillatory solution inside (i.e. the $J_M(\rho)$ function) with an exponentially damped solution outside the fiber (i.e. the $K_M(\rho)$ function). The matching conditions, well known from classical electromagnetic theory, are the continuity conditions for the E_z and H_z field components at the surface of the fiber, when $\rho = a$. Bound states occur because the speed of light is greater in the vacuum than inside the fiber. Therefore, it may happen that $k_\perp$ is real inside and imaginary outside the fiber. Since there are two matching conditions and only one ratio of the amplitudes inside and outside the fiber, both conditions can be satisfied only for a set of discrete eigenvalues of the photon energy $\hbar\omega$. It is worth noting that in order to have an imaginary $k_\perp$ one must have a nonvanishing k_z. Thus, a photon may be bound in the plane perpendicular to the fiber, but it is always moving freely along the fiber, as in the quantum-mechanical description of a charged particle moving in a homogeneous magnetic field. This analysis gives an interpretation of electromagnetic evanescent waves as quantum bound states. Of course, true bound states of photons, that are described by a photon wave function decaying exponentially in *all* directions, are not possible.

§ 7.   Relativistic invariance of photon wave mechanics

In a relativistically invariant quantum theory, the Poincaré transformations are represented by unitary operators. The ten Hermitian generators of these transformations must satisfy the commutation relations characteristic of the Poincaré group. The ten generators of the Poincaré group are identified with the operators $\hat{H}, \hat{\mathbf{P}}, \hat{\mathbf{J}}$, and $\hat{\mathbf{K}}$. They generate infinitesimal time translation, space translations, rotations, and boosts (special Lorentz transformations), respectively. The structure of the Poincaré group leads to the following commutation relations obeyed by these generators (cf., for example, Bargmann and Wigner [1948], Bialynicki-Birula and Bialynicka-Birula [1975], Itzykson and Zuber [1980], Weinberg [1995],

p. 61)

$$\left[\hat{J}_i, \hat{P}_j\right] \;=\; i\hbar\epsilon_{ijk}\hat{P}_k, \tag{7.1}$$

$$\left[\hat{J}_i, \hat{J}_j\right] \;=\; i\hbar\epsilon_{ijk}\hat{J}_k, \tag{7.2}$$

$$\left[\hat{J}_i, \hat{K}_j\right] \;=\; i\hbar\epsilon_{ijk}\hat{K}_k, \tag{7.3}$$

$$\left[\hat{K}_i, \hat{P}_j\right] \;=\; i\hbar c^{-2}\delta_{ij}\hat{H}, \tag{7.4}$$

$$\left[\hat{K}_i, \hat{H}\right] \;=\; i\hbar\hat{P}_i, \tag{7.5}$$

$$\left[\hat{K}_i, \hat{K}_j\right] \;=\; -ic^{-2}\hbar\epsilon_{ijk}\hat{J}_k. \tag{7.6}$$

All the remaining commutators vanish. One may check by a direct calculation that the operators $\hat{H}$, $\hat{\mathbf{P}}$, $\hat{\mathbf{J}}$, and $\hat{\mathbf{K}}$, given in momentum representation by the formulas (5.11)–(5.14) and in coordinate representation by the formulas (5.23)–(5.26), obey the commutation relation for the generators of the Poincaré group. In the proof of the commutation relations in momentum representation one needs the condition (4.19). Since all generators of the Poincaré group are represented by operators that are Hermitian with respect to the scalar product (5.3) or (5.7), the Poincaré transformations are represented by unitary operators. Therefore, the scalar product is invariant under these transformations and all transition probabilities are the same for all observers connected by Poincaré transformations. Thus, in both coordinate and momentum representations, wave mechanics of photons is a fully relativistic theory.

§8. Localizability of photons

The problem of localization of relativistic systems has been first posed and solved by Newton and Wigner [1949] and later refined by Wightman [1962]. According to this analysis it is possible to define position operators and localized states for massive particles and for massless particles of spin 0, but not for massless particles with spin. Thus, the position operator in the sense of Newton and Wigner does not exist for photons (cf. also a recent tutorial review on that subject by Rosewarne and Sarkar [1992]). As a simple heuristic explanation, why position operator for the photon does not exist, one may observe (Pryce [1948]) that the multiplication by $\mathbf{r}$ can not be applied to the photon wave function because it destroys the divergence condition (1.6).

A weaker definition of localization that is applicable even when the position operator does not exist, was proposed by Jauch and Piron [1967] and a very detailed analysis of this problem has been given by Amrein [1969]. The Jauch-Piron localizability allows for noncompatibility of "photon position measurements" in overlapping regions. The main weakness of such an abstract analysis is that an operational definition of the photon position measurement for photons has not been incorporated into it. The existence of position measurements for photons is just taken for granted regardless of the feasibility of their physical realizations. When a realistic model of the photon detector is brought in, it is the wave function Ψ rather than Φ that appears as the correct probability amplitude for photodetection (Mandel and Wolf [1995]). Thus, in practical applications the energy wave functions Ψ always seem to play a dominant role.

It must, however, be stressed that even for massive particles, the localization is not perfect, because it is not relativistically invariant. Two observers who are in relative motion would not quite agree as to the localization region of a relativistic particle. This follows from the fact that the Newton-Wigner wave function ψ^{NW} is related to the relativistic wave function ψ that transforms locally under Poincaré transformations by a nonlocal transformation (cf. Haag [1993])

$$\psi^{NW}(\mathbf{r}) = \int d^3 r' K(\mathbf{r} - \mathbf{r}')\psi(\mathbf{r}'), \tag{8.1}$$

where the kernel K can be represented in terms of the Macdonald function

$$K(\mathbf{r}) = \sqrt{\frac{\pi}{2}}(\frac{2mc}{r\hbar})^{5/4}K_{5/4}(mcr/\hbar). \tag{8.2}$$

In the limit, when $m \to 0$, $K(\mathbf{r}) \to \pi/(2\pi r)^{-5/2}$ and (8.1) becomes the relation (1.2) between the local wave function of the photon and the Landau-Peierls wave function. Thus, the difference between the localizability of massive particles and photons is not that great. In both cases, localization can not be defined in a relativistic manner. However, for massive particles departures from strict localization are only exponentially small due to the fast decay of the Macdonald function in eq. (8.1). In the nonrelativistic limit, when $c \to \infty$, the exponential tails become infinitely sharp and the localization is restored.

Difficulties with the position operator for relativistic particles have a profound origin connected with the structure of the Poincaré group. In nonrelativistic physics the position operator is the generator (up to a factor of mass) of Galilean transformations (cf., for example, Gottfried [1966], Weinberg [1995], p. 62). In a relativistic theory, the Galilean transformations are replaced by the

Lorentz transformations and the position operator (multiplied by the mass) is replaced by the boost generator $\mathbf{K}$. The main difference between Galilean and Lorentz transformation affecting the discussion of localizability is that boost generators *do not commute*. Therefore, one may only hope to localize relativistic particles in one direction at a time. The possibility to localize photons in one direction has been discussed in general terms as the "front" description by Acharya and Sudarshan [1960]. The eigenfunctions of the boost operator K_z given in §6 may serve as an explicit realization of the front description for the photon.

The considerations of photon localizability, while important for the understanding of some fundamental issues, do not influence much the practical applications of the photon wave function. All that really should matter there is that the wave function be precisely defined and that its interpretation be not extended beyond the limits of applicability.

§9.   Phase-space description of a photon

Distribution functions in phase space are a very convenient tool in the description of statistical properties and the study of the classical limit of wave mechanics. A direct analog of the Wigner function (Wigner [1932]) introduced in wave mechanics may also be introduced for photons with the help of the photon wave function. This is done by Fourier transforming the product of the wave function and its complex conjugate. Fourier transforms of the electromagnetic fields similar to the Wigner function have been introduced in optics, first by Walther [1968] in the two-dimensional context of radiative transfer theory and then by Wolf [1976] and by Sudarshan [1979, 1981a,b] in the three-dimensional case. In these papers phase-space distribution functions were defined for the *stationary states* of the electromagnetic field only and they were treated as functions of the frequency. The time-dependent distribution functions can be defined (Bialynicki-Birula [1994]) with the use of the time-dependent wave function. The only formal difference between the standard definition of the Wigner function in nonrelativistic wave mechanics of massive particles and the case of photons is the presence of vector indices. Thus, the photon distribution function in phase space is not a single scalar function but rather a 6×6 Hermitian matrix defined as follows

$$W_{ab}(\mathbf{r}, \mathbf{k}, t) = \int d^3s \, e^{-i\mathbf{k}\cdot\mathbf{s}} \, \Psi_a(\mathbf{r} + \mathbf{s}/2, t)\Psi_b^*(\mathbf{r} - \mathbf{s}/2, t). \qquad (9.1)$$

Similar multi-component distribution functions arise also for a Dirac particle and one can use some of the techniques developed by Bialynicki-Birula Górnicki and Rafelski [1991] to deal with such functions.

Every 6×6 Hermitian matrix can be written in the following block form

$$W_{ab} = \begin{bmatrix} W_{ij}^0 + W_{ij}^3 & W_{ij}^1 - iW_{ij}^2 \\[2mm] W_{ij}^1 + iW_{ij}^2 & W_{ij}^0 - W_{ij}^3 \end{bmatrix}, \tag{9.2}$$

where all 3×3 matrices W_{ij}^α are Hermitian. This decomposition can also be expressed in terms of the ρ matrices

$$W_{ab} = \rho_0 W_{ij}^0 + \rho_1 W_{ij}^1 + \rho_2 W_{ij}^2 + \rho_3 W_{ij}^3. \tag{9.3}$$

The vector indices i and j refer to the components within the upper and lower parts of the wave function and the matrices ρ act on these parts as a whole. The most general photon distribution function, as seen from this analysis, is quite complicated. In general, when the medium induces mixing of the two polarization states, all components of the distribution function are needed. However, when photons propagate in free space, only a subset of these components is sufficient to account for the dynamical properties of photon beams. The simplest case is that of a given helicity. A more interesting case is that of an unpolarized photon beam: a mixture of both helicities with equal weights. This state has to be described by the distribution function because a mixed state can not be treated by pure Maxwell's theory. In all these cases phase-space dynamics can be described by a 3×3 Hermitian matrix, i.e., by just one scalar function and one vector function. To this end, one may introduce the following reduced distribution function

$$W_{ij}(\mathbf{r}, \mathbf{k}, t) = \int d^3 s \, e^{-i\mathbf{k}\cdot\mathbf{s}} \, \psi_i(\mathbf{r} + \mathbf{s}/2, t)\psi_j^*(\mathbf{r} - \mathbf{s}/2, t), \tag{9.4}$$

where ψ_i are the upper or the lower components of the original wave function. The Hermitian matrix W_{ij}^α can be decomposed into a real symmetric tensor and a real vector according to the formula

$$W_{ij}^\alpha = w_{ij}^\alpha + \frac{c}{2i}\epsilon_{ijk}u_k^\alpha. \tag{9.5}$$

The tensor corresponds to the symmetric part of W_{ij}^α and the vector corresponds to the antisymmetric part. The factor of c has been separated out in the second

term since the vector $\mathbf{u}$ is related to the momentum density, while the trace of w_{ij} is related to the energy density (Bialynicki-Birula [1994]).

The equations satisfied by the components of the photon distribution function in free space can be obtained from Maxwell's equations (1.5) and (1.6) for the vector $\mathbf{F}$,

$$\partial_t W_{ij} = c\left(\mathbf{k} + \frac{i}{2}\nabla\right)_m \epsilon_{mil}\, W_{lj} - c\left(\mathbf{k} - \frac{i}{2}\nabla\right)_m W_{il}\, \epsilon_{mlj}, \qquad (9.6)$$

$$\left(\mathbf{k} + \frac{i}{2}\nabla\right)_i W_{ij} = 0 = \left(\mathbf{k} - \frac{i}{2}\nabla\right)_j W_{ij}. \qquad (9.7)$$

This leads to the following set of coupled evolution equations for the real components w_{ij} and u_i

$$\partial_t w_{ij} = -c\epsilon_{ilk}k_l w_{kj} - c\epsilon_{jlk}k_l w_{ki} - \frac{c^2}{2}(\nabla_i u_j + \nabla_j u_i - \delta_{ij}\nabla_k u_k), \qquad (9.8)$$

$$\partial_t u_i = -c\epsilon_{ijk}k_j u_k - \frac{1}{2}(\nabla_j w_{ij} - \nabla_i w_{jj}), \qquad (9.9)$$

and to the subsidiary conditions

$$c\epsilon_{ijk}k_j u_k = \nabla_j w_{ij}, \qquad (9.10)$$

$$c\epsilon_{ijk}\nabla_j u_k = 4k_j w_{ij}. \qquad (9.11)$$

The $\mathbf{k}$-dependent terms in the evolution equations describe a uniform rotation of the vector u_i and of the tensor w_{ij} around the wave vector $\mathbf{k}$ so that these terms can be eliminated by "going to a rotating coordinate frame".

With the help of the subsidiary conditions (9) one can eliminate the remaining components and obtain from the evolution equations (9) the equations for $w = \sum w_{ii}$ and $\mathbf{u}$

$$\partial_t w \;\; = \;\; -c^2 \nabla_i u, \qquad (9.12)$$

$$\partial_t u_i \;\; = \;\; -2c\epsilon_{ijk}k_j u_k - \nabla_i w. \qquad (9.13)$$

These evolution equations do form a simple, self-contained set. However, as it is always the case with the phase-space distribution functions in wave mechanics, not all solutions of these equations are admissible. Only those distribution functions are allowed that can be represented in the form (9.4) at the initial time (with $\mathbf{u}$ and w satisfying the subsidiary conditions (9.10) and (9.11)).

§10.   Hydrodynamic formulation

It has been shown by Madelung [1926] that the Schrödinger wave equation can be cast into a hydrodynamic form. In this form, the complex wave function is replaced by real variables: the probability density ρ and the velocity $\mathbf{u}$ of the probability flow. The wave equation is replaced by the hydrodynamic evolution equations for the variables ρ and $\mathbf{u}$. In order to reduce the number of functions from four to the original two, one has to impose auxiliary conditions — the quantization condition — on the velocity field. Later, other wave equations in quantum mechanics (Pauli, Dirac, Weyl) were also presented in the hydrodynamic form. The wave equation for the photon wave function is not an exception in this respect. It can also be written (Bialynicki-Birula [1996b]) as a set of equations for real hydrodynamic-like variables. Since the Riemann-Silberstein vector $\mathbf{F}$ carries all the information about the photon wave function, one may use $\mathbf{F}$ to define these variables. They comprise the energy density ρ and the velocity of the energy flow $\mathbf{v}$,

$$\rho(\mathbf{r},t) = \mathbf{F}^*(\mathbf{r},t) \cdot \mathbf{F}(\mathbf{r},t), \;\; \rho(\mathbf{r},t)\mathbf{v}(\mathbf{r},t) = \frac{c}{2i}\mathbf{F}^*(\mathbf{r},t) \times \mathbf{F}(\mathbf{r},t), \qquad (10.1)$$

the components t_{ij} of the following tensor

$$t_{ij}(\mathbf{r},t) = F_i^*(\mathbf{r},t)F_j(\mathbf{r},t) + F_j^*(\mathbf{r},t)F_i(\mathbf{r},t), \qquad (10.2)$$

and another vector $\mathbf{u}$,

$$\rho(\mathbf{r},t)\mathbf{u}(\mathbf{r},t) = \frac{c}{2i}\big(\mathbf{F}^*(\mathbf{r},t)\nabla\mathbf{F}(\mathbf{r},t) - (\nabla\mathbf{F}^*(\mathbf{r},t))\mathbf{F}(\mathbf{r},t)\big). \qquad (10.3)$$

Owing to the existence of the following identities satisfied by the hydrodynamic variables

$$t_{ii} = 2c, \quad v_i t_{ik} = 0, \quad t_{ij}t_{ij} = 4c^2 - 2\vec{v}^2, \qquad (10.4)$$

only one component of t_{ij} is arbitrary, but the hydrodynamic equations look more symmetric when all the components are treated on equal footing. The number of algebraically independent hydrodynamic variables is reduced from eight to six (the number of degrees of freedom described by $\mathbf{F}$) by the following quantization condition

$$\int d\vec{S}\cdot\big[\nabla \times \vec{u} - \frac{1}{8c^3}\varepsilon_{ijk}(v_i\nabla v_j \times \nabla v_k + v_i\nabla t_{jl} \times \nabla t_{kl} - 2t_{il}\nabla t_{jl} \times \nabla v_k)\big] = 2\pi n, \tag{10.5}$$

where n is a natural number. This condition must hold for every choice of the integration surface and it states, in essence, that the phase of the wave function is uniquely defined (up to an overall constant phase).

The evolution equations for the hydrodynamic variables are

$$\partial_t\rho + (\vec{v}\cdot\nabla)\rho = -\rho(\nabla\cdot\vec{v}), \qquad (10.6)$$

$$\partial_t v_i + (\vec{v}\cdot\nabla)v_i = \frac{1}{\rho}\partial_j(-c^2\rho\delta_{ij} + \rho v_i v_j + \rho t_{ij}), \qquad (10.7)$$

$$\begin{aligned}
\partial_t t_{ij} + (\vec{v}\cdot\nabla)t_{ij} = {} & \frac{1}{\rho}\Big(t_{ij}v_k\partial_k\rho - c\delta_{ij}v_k\partial_k\rho + \frac{c}{2}(v_i\partial_j + v_j\partial_i)\rho\Big) \\
& + \; \delta_{ij}v_k\partial_l t_{kl} + 2v_k\partial_k t_{ij} + c\varepsilon_{ikl}u_k t_{lj} + c\varepsilon_{jkl}u_k t_{li} \\
& + \; (v_k\partial_k t_{ij} - t_{ij}\partial_k v_k) + \frac{1}{2}(t_{ik}\partial_k v_j + t_{jk}\partial_k v_i) \\
& - \; \frac{1}{2}(v_i\partial_k t_{kj} + v_j\partial_k t_{ki} + v_k\partial_i t_{kj} + v_k\partial_j t_{ki}), \qquad (10.8)
\end{aligned}$$

$$\partial_t u_i + (\vec{v}\cdot\nabla)u_i = \frac{1}{4c\rho}\partial_j\big[\rho\varepsilon_{jkl}(t_{km}\partial_i t_{ml} + v_k\partial_i v_l)\big]. \qquad (10.9)$$

They must be supplemented by the equations that express the divergence condition (1.6)

$$\frac{1}{2}\partial_k(\rho t_{ik}) + \rho\varepsilon_{ijk}v_j u_k \qquad (10.10)$$

$$+ \; \frac{\rho}{4c}(t_{jk}\partial_k t_{ij} - t_{ij}\partial_k t_{jk} + v_k\partial_k v_i - v_i\partial_k v_k) = 0,$$

$$\frac{1}{2}\partial_k(\rho\varepsilon_{ikl}v_l) + \rho t_{ik}u_k \qquad (10.11)$$

$$+ \; \frac{\rho}{4c}\big[\varepsilon_{jkl}(t_{il}\partial_k v_j - v_j\partial_k t_{il}) + \varepsilon_{ijl}(t_{kl}\partial_k v_j - v_j\partial_k t_{kl})\big] = 0.$$

Hydrodynamic description of the photon dynamics is not simple but its existence underscores again the unification of the quantum theory of the photon with the rest of quantum mechanics. Everything that can be done with other particles, can be also done for the photons.

§11. Photon wave function in non-Cartesian coordinate systems and in curved space

It has been shown by Skrotskiĭ [1957] and Plebanski [1960] (for a pedagogical review, see Schleich and Scully [1984]) that the propagation of the electromagnetic field in arbitrary coordinate systems including also the case of curved

spacetime may be described by Maxwell's equations, with all the information about the spacetime geometry contained in the relations connecting the field vectors $\mathbf{E}, \mathbf{B}$ and $\mathbf{D}, \mathbf{H}$. This discovery can be further enhanced by the observation (Bialynicki-Birula [1994]) that in contradistinction to the case of an inhomogeneous medium, in the gravitational field the two photon helicities do not mix. This follows from the fact that for arbitrary metric $g_{\mu\nu}$ the constitutive relations can be written as a single equation connecting two complex vectors: the vector $\mathbf{F}(\mathbf{r}, t)$ defined by the formula (1.7) and a new vector $\mathbf{G}(\mathbf{r}, t)$ defined as

$$\mathbf{G}(\mathbf{r}, t) = \frac{1}{\sqrt{2}} \left(\frac{\mathbf{E}(\mathbf{r}, t)}{\sqrt{\mu_0}} + i \frac{\mathbf{H}(\mathbf{r}, t)}{\sqrt{\epsilon_0}} \right). \tag{11.1}$$

In curved space (or in curvilinear coordinates), the constitutive relations for two complex vectors $\mathbf{F}(\mathbf{r}, t)$ and $\mathbf{G}(\mathbf{r}, t)$ have the form

$$F^i = -\frac{1}{g_{00}} \left(\sqrt{-g} g^{ij} + i g_{0k} \varepsilon^{ikj} \right) G_j, \tag{11.2}$$

$$G_i = -\frac{1}{g^{00}} \left(g_{ij}/\sqrt{-g} - i g^{0k} \varepsilon_{ikj} \right) F^j, \tag{11.3}$$

where $g_{\mu\nu}$ is the metric tensor, $g^{\mu\nu}$ is its inverse, and g is the determinant of $g_{\mu\nu}$. The Maxwell equations expressed in terms of vectors $\mathbf{G}$ and $\mathbf{F}$ in curved spacetime are the same as in flat space

$$i\partial_t \mathbf{F}(\mathbf{r}, t) = \nabla \times \mathbf{G}(\mathbf{r}, t), \tag{11.4}$$
$$\nabla \cdot \mathbf{F}(\mathbf{r}, t) = 0, \tag{11.5}$$

In these equations, all derivatives are *ordinary* (not covariant) derivatives as in flat space. The whole difference is in the form of the constitutive relations (11.2) and (11.3). These relations contain all information about the gravitational field or the curvilinear coordinate system. Since the relations between $\mathbf{G}$ and $\mathbf{F}$ are linear, one may write again two separate wave equations for the two helicity states as in flat space. By combining these two equations, one obtains the following wave equation for the six-component photon wave function $\mathcal{F}(\mathbf{r}, t)$

$$i\partial_t \mathcal{F}(\mathbf{r}, t) = \rho_3 \nabla \times \mathcal{G}(\mathbf{r}, t), \tag{11.6}$$

where

$$\mathcal{G}_i = -\frac{1}{g^{00}} \left(g_{ij}/\sqrt{-g} - i\rho_3 g^{0k} \varepsilon_{ikj} \right) \mathcal{F}^j. \tag{11.7}$$

These equations contain only the matrix ρ_3 that does not mix the helicity states.

The true photon wave function $\Psi(\mathbf{r}, t)$ may be introduced only in the time-independent case, when the separation of $\mathcal{F}(\mathbf{r}, t)$ into positive and negative energy parts is well defined.

§12. Photon wave function as a spinor field

Soon after the formulation of spinor calculus by van der Waerden in the context of the Dirac equation, it has been discovered by Laporte and Uhlenbeck [1931] that the Maxwell equations can also be cast into a spinorial form. The spinor representation of the electromagnetic field and the Riemann-Silberstein vector are closely connected. The components of the vector $\mathbf{F}$ are related to the components of a second rank symmetric spinor ϕ_{AB}

$$\phi_{00} = -F_x + iF_y, \tag{12.1}$$
$$\phi_{01} = F_z, \tag{12.2}$$
$$\phi_{11} = F_x + iF_y, \tag{12.3}$$

and the components of the complex conjugate vector $\mathbf{F}^*$ are related to the components of a second rank symmetric primed spinor $\phi^{A'B'}$,

$$\phi^{0'0'} = -F_x^* - iF_y^*, \tag{12.4}$$
$$\phi^{0'1'} = -F_z^*, \tag{12.5}$$
$$\phi^{1'1'} = F_x^* - iF_y^*. \tag{12.6}$$

The property that even in curved space both helicities propagate without mixing is in the spinorial formalism a simple consequence of the fact that both spinors ϕ_{AB} and $\phi^{A'B'}$ satisfy separate wave equations (Laporte and Uhlenbeck [1931], Penrose and Rindler [1984])

$$\sigma^{\mu C'A}\partial_\mu \phi_{AB}(\mathbf{r}, t) = 0, \tag{12.7}$$
$$\sigma^\mu{}_{CA'}\partial_\mu \phi^{A'B'}(\mathbf{r}, t) = 0, \tag{12.8}$$

where the matrices $\sigma^{\mu A'B}$ and $\sigma^\mu{}_{A'B}$ are built from the unit matrix and the Pauli matrices,

$$(\sigma^{\mu A'B}) = (I, \boldsymbol{\sigma})^{A'B}, \tag{12.9}$$
$$(\sigma^\mu{}_{AB'}) = (I, -\boldsymbol{\sigma})_{AB'}. \tag{12.10}$$

Under a Lorentz transformation the second rank spinor changes according to the formula

$$\phi'_{AB}(\mathbf{r}',t') = S_A{}^C S_B{}^D \phi_{CD}(\mathbf{r},t), \tag{12.11}$$

where $S_A{}^C$ is a 2×2 matrix of the fundamental (spinor) representation of the Lorentz group. Thus, from the point of view of the representation theory of the Lorentz and Poincaré groups (Streater and Wightman [1978], Weinberg [1995], p. 231), the photon wave functions for a given helicity are just the three-component fields that transform as irreducible representations $(1,0)$ or $(0,1)$ of the proper Lorentz group (without reflections). In order to accommodate reflections one must combine both representations and introduce the six-dimensional objects $\mathcal{F}$ or Ψ.

Eqs. (12.7) (and similarly eqs. (12.8)) represent a set of four equations obeyed by four components $(\phi_{11}, \phi_{12}, \phi_{21}, \phi_{22})$ of the second order spinor. All four equations can be written in the form of the Dirac equation (Ohmura [1956], Moses [1959])

$$i\hbar\partial_t\phi(\mathbf{r},t) = \boldsymbol{\alpha}\cdot\frac{\hbar}{i}\nabla\phi(\mathbf{r},t), \tag{12.12}$$

where the matrices $\boldsymbol{\alpha}$ are

$$\alpha_x = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \alpha_y = \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix}, \quad \alpha_z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{12.13}$$

One may check that in this formulation the divergence condition takes on the following simple algebraic form: $\phi_{12} = \phi_{21}$.

The Maxwell equations expressed in spinor notation and the Weyl equation provide just the simplest examples from a hierarchy of wave equations for massless fields described by symmetric spinors $\phi_{B_1 B_2 \cdots B_n}$ or $\phi^{B'_1 B'_2 \cdots B'_n}$. All these equations have the form (Penrose and Rindler [1984])

$$\sigma^{\mu C' A}\nabla_\mu\phi_{AB_1 B_2 \cdots B_{n-1}}(\mathbf{r},t) = 0, \tag{12.14}$$

$$\sigma^\mu{}_{CA'}\nabla_\mu\phi^{A'B'_1 B'_2 \cdots B'_{n-1}}(\mathbf{r},t) = 0. \tag{12.15}$$

This universality of massless wave equations for all spins gives an additional argument for treating the Riemann-Silberstein vector as the photon wave function.

§13. Photon wave functions and mode expansion of the electromagnetic field

The concept of the photon wave function is also useful in the process of quantization of the electromagnetic field. One may simply apply the procedure of second quantization to the photon wave function in the same manner as one does it for other field operators. In order to see this analogy, one may recall that the field operator $\hat{\psi}(\mathbf{r})$ for, say the electron field, is built from a complete set of wave functions for the electrons $\psi_n^+(\mathbf{r})$ and from a complete set of wave functions for positrons $\psi_n^-(\mathbf{r})$ according to the following rule (cf., for example, Schweber [1961], Bialynicki-Birula and Bialynicka-Birula [1975], Weinberg [1995]

$$\hat{\psi}(\mathbf{r}, t) = \sum_n (\psi_n^+(\mathbf{r}, t)\hat{a}_n + \psi_n^-(\mathbf{r}, t)\hat{b}_n^\dagger), \tag{13.1}$$

where $\hat{a}_n$ and $\hat{b}_n$ are the annihilation operators for electrons and positrons respectively. The second part of the field operator is related to the first by the operation of charge conjugation performed on the wave functions and on the operators. The analog of charge conjugation for photon wave functions is given by eq. (3.2). Following this procedure, one may construct the field operator of the electromagnetic field in the form

$$\hat{\mathcal{F}}(\mathbf{r}, t) = \sum_n (\Psi_n(\mathbf{r}, t)\hat{c}_n + \rho_1 \Psi_n^*(\mathbf{r}, t)\hat{c}_n^\dagger), \tag{13.2}$$

where the identity of particles and antiparticles for photons has been taken into account by using only one set of creation and annihilation operators. The field operator (13.2) is non-Hermitian but it satisfies the particle-antiparticle conjugation condition (3.2). Therefore, it only has six Hermitian components. These Hermitian operators are identified as the field operators $\hat{\mathbf{D}}(\mathbf{r}, t)$ and $\hat{\mathbf{B}}(\mathbf{r}, t)$ and they are obtained from $\hat{\mathcal{F}}$ through the formula

$$\begin{bmatrix} \hat{\mathbf{D}} \\ \hat{\mathbf{B}} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \hat{\mathcal{F}}. \tag{13.3}$$

As a direct consequence of the formula (13.2) one may identify the photon wave functions in the second-quantized theory with the matrix elements of the electromagnetic field operators $\hat{\mathcal{F}}$ or $\hat{\mathbf{D}}$ and $\hat{\mathbf{B}}$ taken between one-particle states and the vacuum

$$\Psi_n(\mathbf{r}, t) = \langle 0|\hat{\mathcal{F}}(\mathbf{r}, t)\hat{c}_n^\dagger|0\rangle. \tag{13.4}$$

In the simplest case of the free field, when the complete set of photon wave functions may be labeled by the wave vector $\mathbf{k}$ and helicity λ, the formula (13.2) takes on the form

$$\hat{\mathcal{F}}(\mathbf{r}, t) = \sqrt{\hbar c} \int \frac{d^3 k}{(2\pi)^3} \begin{bmatrix} \mathbf{e}(\mathbf{k}, 1)(\hat{c}(\mathbf{k}, 1)e^{-i\omega t + i\mathbf{k}\cdot\mathbf{r}} + \hat{c}^\dagger(\mathbf{k}, -1)e^{i\omega t - i\mathbf{k}\cdot\mathbf{r}}) \\ \mathbf{e}(\mathbf{k}, -1)(\hat{c}(\mathbf{k}, -1)e^{-i\omega t + i\mathbf{k}\cdot\mathbf{r}} + \hat{c}^\dagger(\mathbf{k}, 1)e^{i\omega t - i\mathbf{k}\cdot\mathbf{r}}) \end{bmatrix}. \quad (13.5)$$

In the presence of a medium, the expansion (13.2) of the electromagnetic field operator requires the knowledge of a complete set of wave functions Ψ_n that satisfy the photon wave equation in the medium. These functions are called usually the mode functions of the electromagnetic field (cf., for example, Louisell [1973] p. 240 and Mandel and Wolf [1995] p. 905) but the term photon wave functions is perhaps more appropriate (Moses [1973], Bialynicki-Birula and Bialynicka-Birula [1975]. The advantage of using the terminology of wave functions is that it brings in all the associations with wave mechanics and makes the classification of the modes more transparent. In particular, one may use the quantum-mechanical notion of eigenfunctions and eigenvalues to classify the functions used in the mode expansion (Moses [1973]) and also borrow from quantum mechanics the methods of proving their completeness (Bialynicki-Birula and Brojan [1972]).

This discussion shows that the photon wave function is not restricted to the *wave mechanics* of photons. The same wave functions also appear as the mode functions in the expansion of the electromagnetic field operators.

§14. Summary

The aim of this review was to collect and explain all basic properties of a certain well defined mathematical object — a six-component function of space-time variables — that describes the quantum state of the photon. Whether one decides to call this object the photon wave function in coordinate representation is a matter of opinion since some properties known from wave mechanics of massive particles are missing. The most essential property that does not hold for the photon wave function is that the argument $\mathbf{r}$ of the wave function can not be directly associated with the position operator of the photon. The position operator for the photon simply *does not exist*. However, one should remember that also for massive particles the true position operator exists only in the nonrelativistic approximation. The concept of localization associated with the Newton-Wigner position operator is *not relativistically invariant*. Since photons can not be described in a nonrelativistic manner, there is no approximate position operator.

The strongest argument that can be made for the photon wave function in coordinate representation is based on the most fundamental property of quantum states — on the *principle of superposition*. According to the superposition principle, wave functions form a linear space. By adding wave functions one obtains again legitimate wave function. Once this principle is accepted, the existence of photon wave functions in coordinate representation follows from the existence of the photon wave functions in momentum representation and these functions are genuine by all standards; their existence simply follows from relativistic quantum kinematics (or more precisely from the representation theory of the Poincaré group). The Fourier integral (4.20) represents a special combination of momentum space wave functions with different momenta and as a matter of principle, such linear combinations are certainly allowed. One may only argue which superpositions to take as more natural or useful but totally rejecting the very concept of the photon wave function in coordinate representation is tantamount to rejecting the superposition principle altogether.

There is not much advantage in using the photon wave function in coordinate representation to perform calculations for photons moving in *free space*. The relation of this wave function to momentum wave function is so straightforward that one may as well stick to momentum representation. It is only in the presence of a medium, especially in an inhomogeneous medium, that the photon wave function in coordinate representation becomes useful and even essential. Only in the coordinate representation one may hope to solve the eigenvalue problems and to take into account the boundary conditions.

The introduction of the wave function for the photon has many significant benefits. The photon wave function enables one to formulate a consistent wave mechanics of photons that could be often used as a convenient tool in the quantum description of electromagnetic fields, *independently* of the formalism of second quantization. In other words, in constructing quantum theories of photons one may proceed, as in quantum theory of all other particles, through two stages. At the first stage one introduces wave functions and a wave equation obeyed by these wave functions. At the second stage one upgrades the wave functions to the level of field operators in order to deal more effectively with the states involving many indistinguishable particles and to allow for processes in which the number of particles is not conserved. Many methods that have proven very useful in the study of particles described by the Schrödinger wave functions can also be implemented for photons leading to some new insights. These methods include relationships between symmetries and operators, the definitions of various sets of modes for the electromagnetic field and their completeness relations, eigenvalue problems for various observables(§6), phase-space

representation (§9), and the hydrodynamic formulation (§10). Finally, there are important logical and pedagogical advantages coming from the use of the photon wave function. The quantum mechanical description of *all* particles, including photons, becomes uniform.

Acknowledgments

§ References

Acharya and Sudarshan [1960]
  Acharya, R., and E.C.G. Sudarshan, 1960, "Front" description in relativistic quantum mechanics, J. Math. Phys. **1**, 532.
Akhiezer and Berestetskii [1965]
  Akhiezer, A.I., and V.B. Berestetskii, 1965, Quantum Electrodynamics, (Interscience, New York), Ch. 1.
Amrein [1969]
  Amrein, W.O., 1969, Localizability for particles of zero mass, Helv. Phys. Acta **42**, 149.
Ashtekar and Magnon [1975]
  Ashtekar, A., and A. Magnon, 1975, Quantum fields in curved space-times, Proc. Roy. Soc. (London) A**346**, 375.
Bargmann and Wigner [1948]
  Bargmann, V., and E.P. Wigner, 1948, Group theoretical discussion of relativistic wave equations, Proc. Natl. Acad. Sci. USA **34**, 211. Reprinted in: Symmetry Groups in Nuclear and Particle Physics, ed. F.J. Dyson (Benjamin, New York) p. 205.
Bateman [1915]
  Bateman, H., 1915, The Mathematical Analysis of Electrical and Optical Wave Motion on the Basis of Maxwell's Equations, Cambridge (reprinted by Dover, New York 1955).
Baym [1969]
  Baym, G., 1969, Lectures on Quantum Mechanics (Benjamin, Reading), Ch. 1.
Bialynicki-Birula [1963]
  Bialynicki-Birula, I., 1963, Quantum electrodynamics without electromagnetic field, Phys. Rev. **130**, 465.

Bialynicki-Birula and Brojan [1972]
   Bialynicki-Birula, I., and J. Brojan, 1972, Completeness of evanescent waves, Phys.
   Rev. D**5**, 485.
Bialynicki-Birula and Bialynicka-Birula [1975]
   Bialynicki-Birula, I., and Z. Bialynicka-Birula, 1975, Quantum Electrodynamics
   (Pergamon, Oxford).
Bialynicki-Birula and Bialynicka-Birula [1976]
   Bialynicki-Birula, I., and Z. Bialynicka-Birula, 1976, Quantum electrodynamics of
   intense photon beams. New approximation method, Phys. Rev. A**14**, 1101.
Bialynicki-Birula and Bialynicka-Birula [1987]
   Bialynicki-Birula, I., and Z. Bialynicka-Birula, 1987, Berry's phase in the relativistic
   theory of spinning particles, Phys. Rev. D **35**, 2383.
Bialynicki-Birula Górnicki and Rafelski [1991]
   Bialynicki-Birula, I., P. Górnicki, and J. Rafelski, 1991, Phase-space structure of
   the Dirac vacuum, Phys. Rev. D**44**, 1825.
Bialynicki-Birula [1994]
   Bialynicki-Birula, I., 1994, On the wave function of the photon, Acta Phys. Polon.
   A**86**, 97.
Bialynicki-Birula [1996a]
   Bialynicki-Birula, I., 1996a, The photon wave function, in: Coherence and Quantum
   Optics VII, eds. J. H. Eberly, L. Mandel, and E. Wolf (Plenum Press, New York).
Bialynicki-Birula [1996b]
   Bialynicki-Birula, I., 1996b, Hydrodynamics of relativistic probability flows, in:
   Nonlinear Dynamics, Chaotic and Complex Systems, eds. E. Infeld, R. Żelazny,
   and A. Gałkowski (Cambridge University, Cambridge).
Bjorken [1963]
   Bjorken, J., 1963, A dynamical origin of the electromagnetic field, Ann. Phys. (New
   York) **24**, 174.
Bohm [1954]
   Bohm, D., 1954, Quantum Theory (Constable, London) p. 91.
Born and Wolf [1980]
   Born, M., and E. Wolf, 1980, Principles of Optics (Pergamon, Oxford) p. 494.
Cohen-Tannoudji, Diu and Laloë [1977]
   Cohen-Tannoudji, C., B. Diu, and F. Laloë, 1977, Quantum Mechanics, Vol. I
   (Wiley, New York).
Cohen-Tannoudji, Dupont-Roc and Grynberg [1989]
   Cohen-Tannoudji, C., J. Dupont-Roc and G. Grynberg, 1989, Photons and Atoms,
   Introduction to Quantum Electrodynamics, John Wiley, New York.
Cook [1982a]
   Cook, R.J., 1982a, Photon dynamics, Phys. Rev. A**25**, 2164.
Cook [1982b]
   Cook, R.J., 1982b, Lorentz covariance of photon dynamics, Phys. Rev. A**26**, 2754.

Dirac [1958]
  Dirac, P.A.M., 1958, The Principles of Quantum Theory 4th ed. (Clarendon Press,
  Oxford) p. 9.
Good [1957]
  Good, R.H.Jr., 1957, Particle aspect of the electromagnetic field equations, Phys.
  Rev. **105**, 1914.
Gottfried [1966]
  Gottfried, K., 1966, Quantum Mechanics, (Benjamin, New York), p. 252.
Gross [1964]
  Gross, L., 1964, Norm invariance of mass-zero equations under the conformal group,
  J. Math. Phys. **5**, 687.
Haag [1993]
  Haag, R., 1993, Local Quantum Physics (Springer, Berlin) p. 33.
Inagaki [1994]
  Inagaki, T., 1994, Quantum-mechanical approach to a free photon, Phys. Rev. A**49**,
  2839.
Itzykson and Zuber [1980]
  Itzykson, C., and J.-B. Zuber, 1980, Quantum Field Theory (McGraw-Hill, New
  York) p. 50.
Jackson [1975]
  Jackson, J.D., 1975, Classical Electrodynamics (Wiley, New York) p. 552.
Jauch and Piron [1967]
  Jauch, J.M., and C. Piron, 1967, Generalized localizability, Helv. Phys. Acta **40**,
  559.
Kaiser [1992]
  Kaiser, G., 1992, Wavelet electrodynamics, Phys. Lett. A**168**, 28.
Kidd, Ardini and Anton [1989]
  Kidd, R., J. Ardini, and A. Anton, 1989, Evolution of the modern photon, Am. J.
  Phys. **57**, 27.
Kramers [1938]
  Kramers, H.A., 1938 Quantentheorie des Elektrons und der Strahlung in: Hand-
  und Jahrbuch der Chemischen Physik, Eucken-Wolf, Leipzig, (English translation:
  Quantum Mechanics (North-Holland, Amsterdam 1957).
Landau and Peierls [1930]
  Landau, L.D., and R. Peierls, 1930, Quantenelektrodynamik im Konfigura-
  tionsraum, Z. Phys. **62**, 188.
Laporte and Uhlenbeck [1931]
  Laporte, O., and G.E. Uhlenbeck, 1931, Application of spinor analysis to the
  Maxwell and Dirac equations. Phys. Rev. **37**, 1380.
Lipkin [1973]
  Lipkin, H.J., 1973, Quantum Mechanics (North-Holland, Amsterdam) Ch. 1.

Lomont and Moses [1962]

Lomont, J.S., and H.E. Moses, 1962, Simple realizations of the infinitesimal generators of the proper orthochronous inhomogeneous Lorentz group for zero mass, J. Math. Phys. **3**, 405.

London [1927]

London, F., 1927, Winkelvariable und kanonische Transformationen in der Undulationsmechanik, Z. Phys. **40**, 193.

Louisell [1973]

Louisell, W.H. 1973, Quantum Statistical Properties of Radiation, (Wiley, New York).

Madelung [1926]

Madelung, E., Quantentheorie in Hydrodynamischer Form, Z. Phys. **40**, 322.

Mandel and Wolf [1995]

Mandel, L., and E. Wolf, 1995, Optical coherence and quantum optics, (Cambridge University Press, Cambridge).

Messiah [1961]

Messiah, A., 1961, Quantum Mechanics, Vol. II (North-Holland, Amsterdam) p. 1034.

Mignani, Recami and Baldo [1974]

Mignani, R., E. Recami, and M. Baldo, 1974, About a Dirac-like equation for the photon according to Ettore Majorana, Lett.Nuovo Cimento **11**, 568.

Misner and Wheeler [1957]

Misner, C.W., and J.A. Wheeler, 1957, Classical physics as geometry: Gravitation, electromagnetism, unquantized charge, and mass as properties of curved empty space, Ann. Phys. **2**, 525.

Molière [1949]

Molière, G., 1949, Laufende elektromagnetische Multipolwellen und eine neue Methode der Feld-Quantisierung, Ann. d. Phys. **6**, 146.

Moses [1959]

Moses, H.E., 1959, Solution of Maxwell's equations in terms of a spinor notation: the direct and inverse problem, Phys. Rev. **113**, 1670.

Moses [1973]

Moses, H.E., 1973, Photon wave functions and the exact electromagnetic matrix elements for hydrogenic atoms, Phys. Rev. A**8**, 1710.

Newton and Wigner [1949]

Newton, T.D., and E.P. Wigner, Localized states for elementary systems, 1949, Rev. Mod. Phys. **21**, 400.

Ohmura [1956]

Ohmura, T., 1956, A new formulation on the electromagnetic field, Prog. Theor. Phys. **16**, 684.

Oppenheimer [1931]
  Oppenheimer, J.R., 1931, Note on light quanta and the electromagnetic field, Phys. Rev. **38**, 725.
Pauli [1933]
  Pauli, W., 1933, Prinzipien der Quantentheorie, in: Handbuch der Physik, Vol. 24 (Springer, Berlin), (English translation: General Principles of Quantum Mechanics (Springer, Berlin, 1980) p. 189.
Pegg and Barnett [1988]
  Pegg, D.T., and S.M. Barnett, Unitary phase operator in quantum mechanics, Europhys. Lett. **6**, 483.
Penrose and Rindler [1984]
  Penrose, R., and W. Rindler, 1984, Spinors and Space-Time, Vol. I (Cambridge University Press, Cambridge) Ch. 5.
Plebanski [1960]
  Plebanski, J., 1960, Electromagnetic waves in gravitational fields, Phys. Rev. **118**, 1396.
Power [1964]
  Power, E.A., 1964, Introductory Quantum Electrodynamics (Longmans, London) p. 61.
Pryce [1948]
  Pryce, M.H.L., 1948, The mass-centre in the restricted theory of relativity and its connection with the quantum theory of elementary particles, Proc. Roy. Soc. (London) A**195**, 62.
Rosewarne and Sarkar [1992]
  Rosewarne, D., and S. Sarkar, 1992, Rigorous theory of photon localizability, Quantum Opt. **4**, 405.
Schiff [1968]
  Schiff, L.I., 1968, Quantum Mechanics (McGraw-Hill, New York) p. 203.
Schleich and Scully [1984]
  Schleich, W., and M.O. Scully, General Relativity and Modern Optics in: New Trends in Atomic Physics, Les Houches, Session XXXVIII, eds. G. Grynberg and R. Stora (Elsevier, Amsterdam 1984) p. 995.
Schweber [1961]
  Schweber, S.S, 1961, An Introduction to Relativistic Quantum Field Theory (Row, Peterson, and Co., Evanston).
Segal [1963]
  Segal, I.E., 1963, Mathematical Problems of Relativistic Physics, (American Mathematical Society, Providence).
Silberstein [1907a]
  Silberstein, L., 1907a, Elektromagnetische Grundgleichungen in bivectorieller Behandlung, Ann. d. Phys. **22**, 579.

Silberstein [1907b]
  Silberstein, L., 1907b, Nachtrag zur Abhandlung über "Elektromagnetische Grund-gleichungen in bivectorieller Behandlung", Ann. d. Phys. **24**, 783.
Silberstein [1914]
  Silberstein, L. 1914, The Theory of Relativity, MacMillan, London (2nd enlarged edition 1924).
Sipe [1995]
  Sipe, J.F., 1995, Photon wave functions, Phys. Rev. A**52**, 1875.
Skrotskiĭ [1957]
  Skrotskiĭ, G. B., 1957, The influence of gravitation on the propagation of light, Soviet Physics Doklady **2**, 226.
Staruszkiewicz [1973]
  Staruszkiewicz, A., 1973, On affine properties of the light cone and their application in quantum electrodynamics, Acta Phys. Polon. B**4**, 57.
Streater and Wightman [1978]
  Streater, R.F., and A.S. Wightman, 1978, PCT, Spin and Statistics, and All That (Benjamin, Reading).
Sudarshan [1979]
  Sudarshan, E.C.G., 1979, Pencils of rays in wave optics, Phys. Lett. A**73**, 269.
Sudarshan [1981a]
  Sudarshan, E.C.G., 1981a, Quantum electrodynamics and light rays, Physica A**96**, 315.
Sudarshan [1981b]
  Sudarshan, E.C.G., 1981b, Quantum theory of radiative transfer, Phys. Rev. A**23**, 2803.
Walther [1968]
  Walther, A., 1968, Radiometry and coherence, J. Opt. Soc. Am. **68**, 1256.
Weber [1901]
  Weber, H., 1901, Die partiellen Differential-Gleichungen der mathematischen Physik nach Riemann's Vorlesungen (Friedrich Vieweg und Sohn, Braunschweig) p. 348.
Weinberg [1995]
  Weinberg, S., 1995, The quantum theory of fields, Vol. I (Cambridge University, Cambridge) p. 61.
Weinberg and Witten [1980]
  Weinberg, S., and E. Witten, 1980, Limits on massless particles, Phys. Lett. B**96**, 59.
Weyl [1929]
  Weyl, H., 1929, Elektron und Gravitation, Z. Phys. **56**, 330.
Wightman [1962]
  Wightman, A.S., 1962, On the localizability of quantum mechanical systems, Rev. Mod. Phys. **34**, 845.

Wigner [1932]
  Wigner, E.P., 1932, On the quantum correction for thermodynamic equilibrium,
  Phys. Rev. **40**, 749.
Wigner [1939]
  Wigner, E.P., 1939, On unitary representations of the inhomogeneous Lorentz group,
  Ann. of Math. **40**, 39. Reprinted: 1966, Symmetry Groups in Nuclear and Particle
  Physics, ed. F. J. Dyson (Benjamin, New York) p. 120.
Wolf [1976]
  Wolf, E., 1976, New theory of radiative energy transfer in free electromagnetic field,
  Phys. Rev. D**13**, 869.
Zeldovich [1965]
  Zeldovich, Ya.B., Number of quanta as an invariant of the classical electromagnetic
  field, Dokl. Acad. Sci. USSR, **163**, 1359 (In Russian).